Fractional Hedonic Games

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Introduction

An important issue in multi-agent systems is the exploitation of synergies via coalition formation. We initiate the formal study of *fractional hedonic games*. In fractional hedonic games, the utility of a player in a coalition structure is the average value he ascribes to the members of his coalition. Among other settings, this covers situations in which there are several types of agents and each agent desires to be in a coalition in which the fraction of agents of his own type is minimal. Fractional hedonic games not only constitute a natural class of succinctly representable coalition formation games, but also provide an interesting framework for network clustering. We propose a number of conditions under which the core of fractional hedonic games is non-empty and provide algorithms for computing a core stable outcome.

Fractional Hedonic Games

A **hedonic game** is a tuple $(N, \succsim_1, \dots, \succsim_n)$, where

- $N = \{1, \dots, n\}$ a set of players
- $\succsim_i$ a transitive and complete preference relation over the coalitions i is a member of.

A **coalition structure** π is a partition of the players into coalitions. By $\pi(i)$ we denote the coalition i is in under coalition structure π.

Question Which coalitions can reasonably be expected to form?

A hedonic game is a **fractional hedonic game** if for each player i there is a **valuation function** $v_i\colon N \to \mathbb{R}$ of the players, such that

$$S \succsim_i T \quad \text{if and only if} \quad \frac{\sum_{j\in S} v_i(j)}{|S|} \geq \frac{\sum_{j\in T} v_i(j)}{|T|}$$

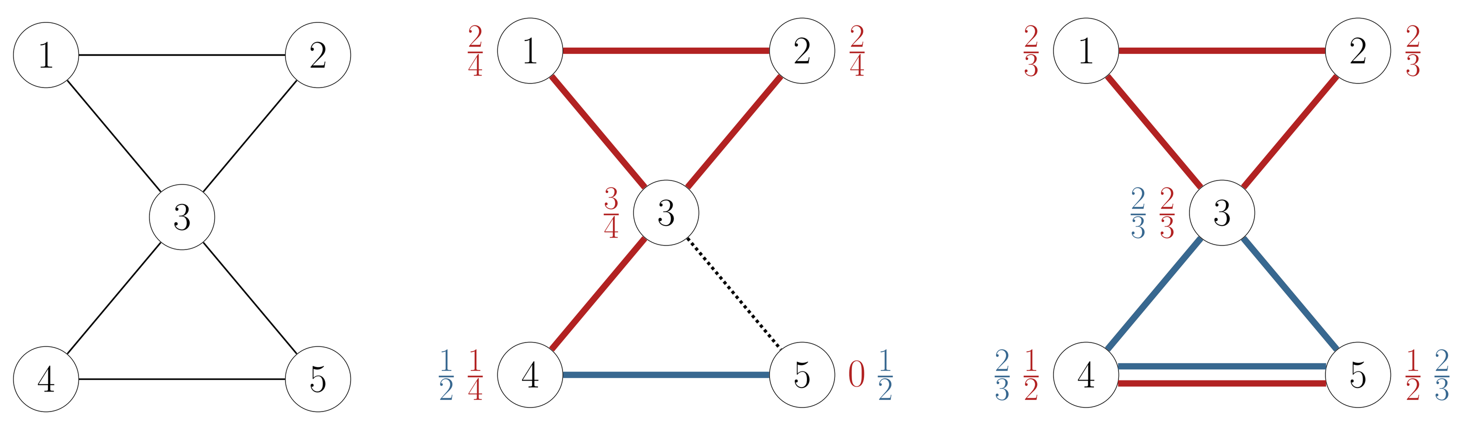
Intuition Each player i has value $v_i(j)$ for player j being in the same coalition. The utility of a coalition S for player i is the average value of its coalition members.

Preferences are **symmetric** if players like/dislike each other with the same intensity. Preferences are **simple** if $v_i(j)$ is either 0 or 1 for all players i and j.

Observation Simple and symmetric fractional hedonic games can naturally be represented by undirected and unweighted graphs, with vertices corresponding to players and an edge $\{i, j\}$ indicating that $v_i(j) = 1$.

Different solution concepts for fractional hedonic games also give rise to desirable partitions which correspond to **desirable clusterings** for a network. We mainly focus on the core and the strict core.

Example



Example of a simple and symmetric fractional hedonic game represented by a graph. The utility of player 2 for coalition $\{1, 2, 3, 4\}$ is $\frac{2}{4}$ and that of player 3 for the same coalition is $\frac{3}{4}$. The partition $\{\{1, 2, 3, 4\}, \{5\}\}$ is not core stable, as it is blocked by $\{4, 5\}$. The partition $\{\{1, 2, 3\}, \{4, 5\}\}$ is core stable, but not strict core stable.

Stable Partitions

A coalition $S \subseteq N$ **strongly blocks** a partition π, if each player $i \in S$ strictly prefers S to $\pi(i)$. A partition which admits no blocking coalition is in the **core**.

A coalition $S \subseteq N$ **weakly blocks** a partition π, if each player $i \in S$ weakly prefers S to $\pi(i)$ and there exists at least one player $j \in S$ who strictly prefers S to his current coalition $\pi(j)$. A partition which admits no weakly blocking coalition is in the **strict core**.

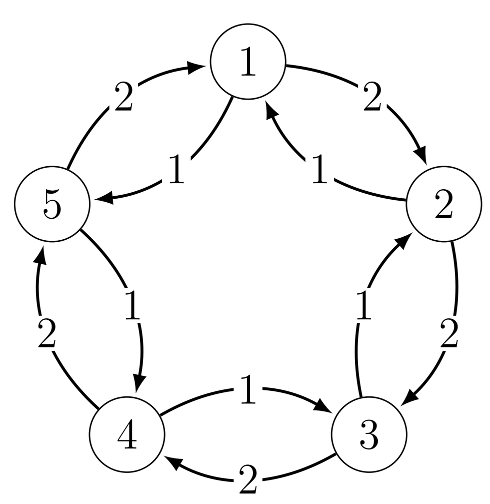
Negative Results

Theorem For fractional hedonic games, the core can be empty.

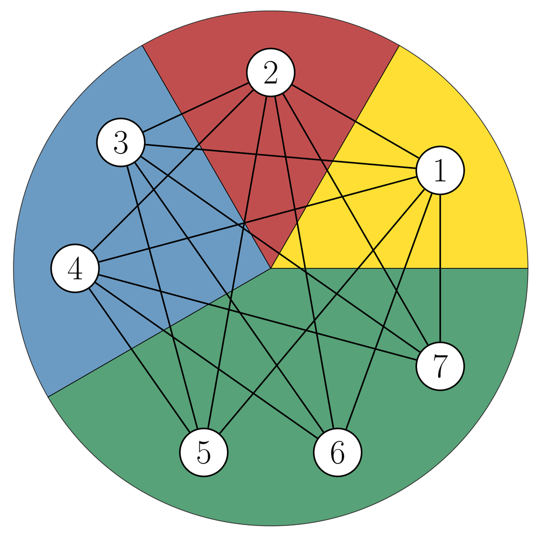
Theorem Computing a core stable partition for fractional hedonic games with symmetric preferences is NP-hard. Moreover, checking whether a partition is core stable is coNP-complete.

Theorem In simple symmetric fractional hedonic games, the strict core can be empty.

Remark It remains open whether simple symmetric fractional hedonic games always admit core stable partitions.



Bakers and Millers



Players are divided into types $\theta_1, \dots, \theta_k$, e.g., bakers and millers. Player i of type $\theta(i)$ prefers to be in a coalition with as many players of other types and as few players of the same type as possible:

$$S \succsim_i T \quad \text{if and only if} \quad \frac{|S \cap \theta(i)|}{|S|} \leq \frac{|T \cap \theta(i)|}{|T|}.$$

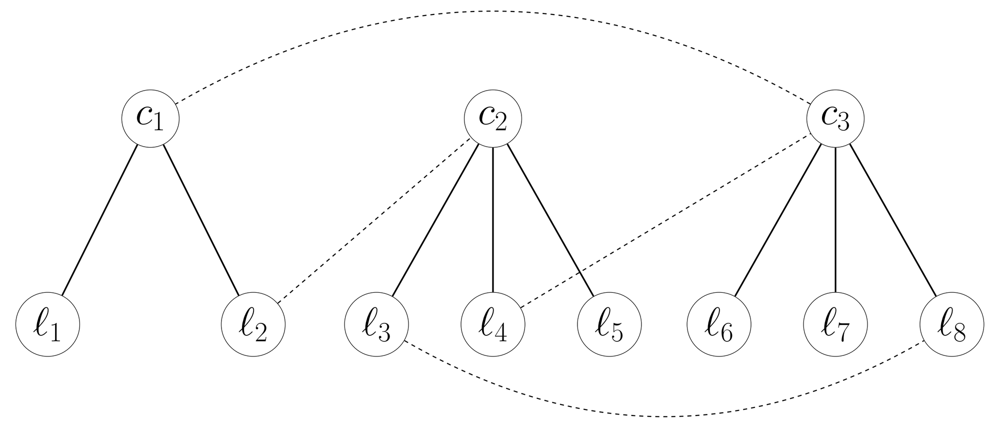
These **Bakers and Millers games** are naturally modelled as complete k-partite graphs.

Theorem A partition π of a Bakers and Millers game is strict core stable if and only if for all types θ and all coalitions $S, S' \in \pi$,

$$\frac{|S \cap \theta|}{|S|} = \frac{|S' \cap \theta|}{|S'|}.$$

Corollary For every Bakers and Millers game, there is a unique finest strict core stable partition (up to renaming players of the same type), which, moreover, can be computed in linear time.

Graphs with Large Girth



Theorem For simple symmetric fractional hedonic games represented by graphs with girth at least five, the core is non-empty.

Other Positive Results

Theorem For simple symmetric fractional hedonic games represented by graphs of degree at most 2, the core is non-empty.

Theorem For simple symmetric fractional hedonic games represented by undirected forests, the core is non- empty.

Lemma For every fractional hedonic game that is represented by an undirected bipartite graph admitting a perfect matching the core is non-empty.

Theorem For all bipartite k-regular graphs the core of the corresponding fractional hedonic game is non-empty.

Observation Most of our positive results imply polynomial-time algorithms for computing the core.

Acknowledgements

This material is supported by the Australian Government's Department of Broadband, Communications and the Digital Economy, the Australian Research Council, the Asian Office of Aerospace Research and Development under grant AOARD-124056, the Deutsche Forschungsgemeinschaft under grants BR 2312/7-1 and BR 2312/10-1, and the ERC under Advanced Grant 291528 ("RACE").

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