Majority Graphs of Assignment Problems and Properties of Popular Random Assignments

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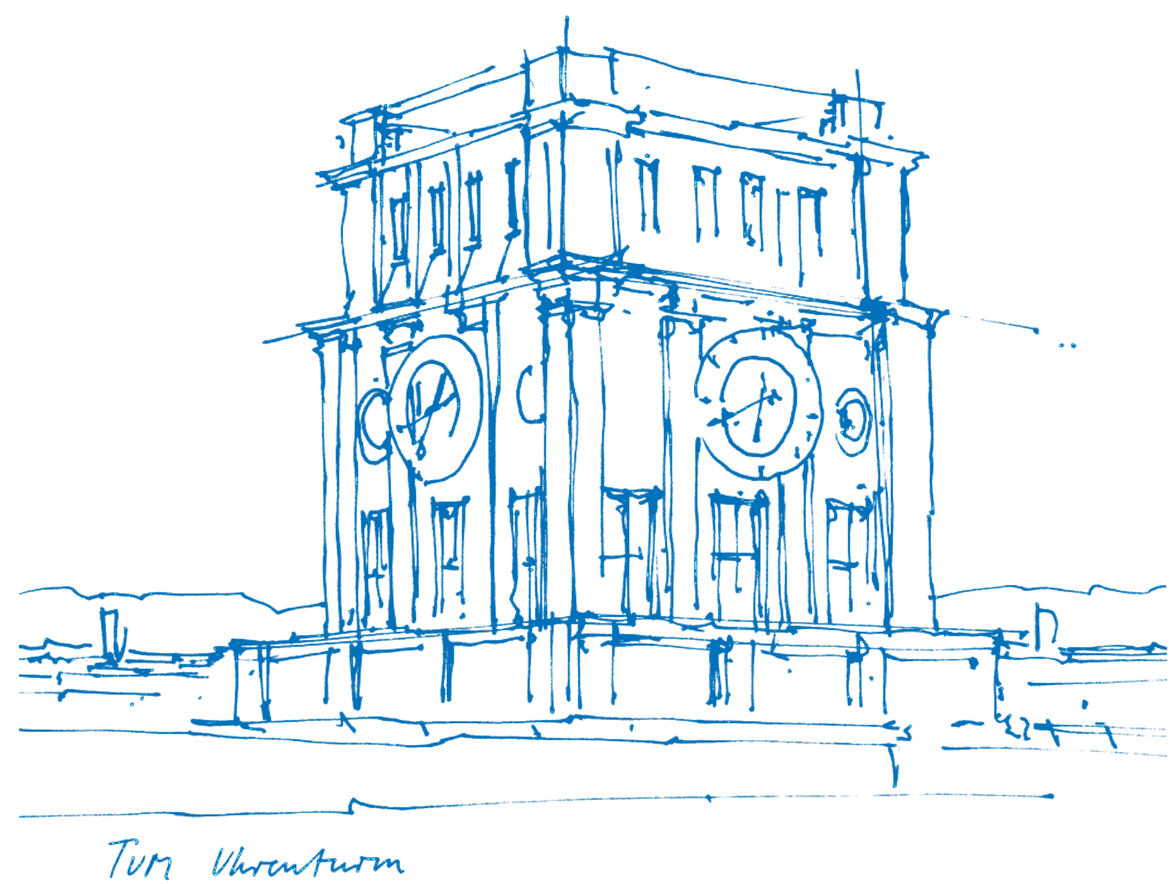
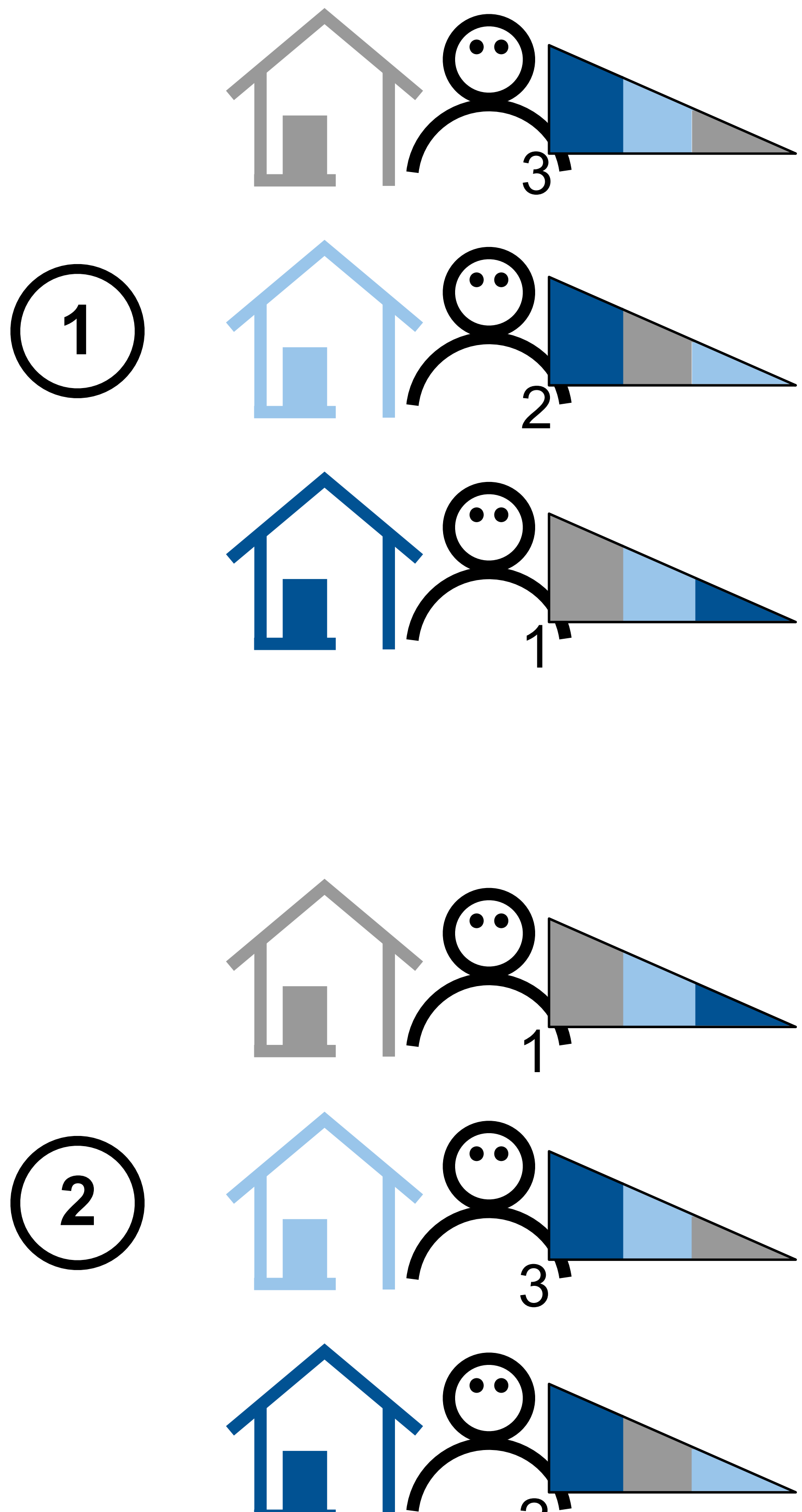
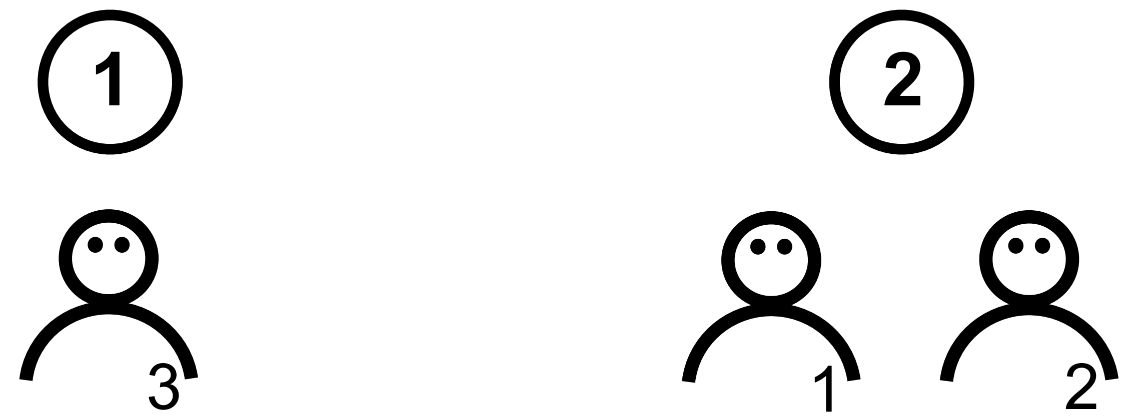




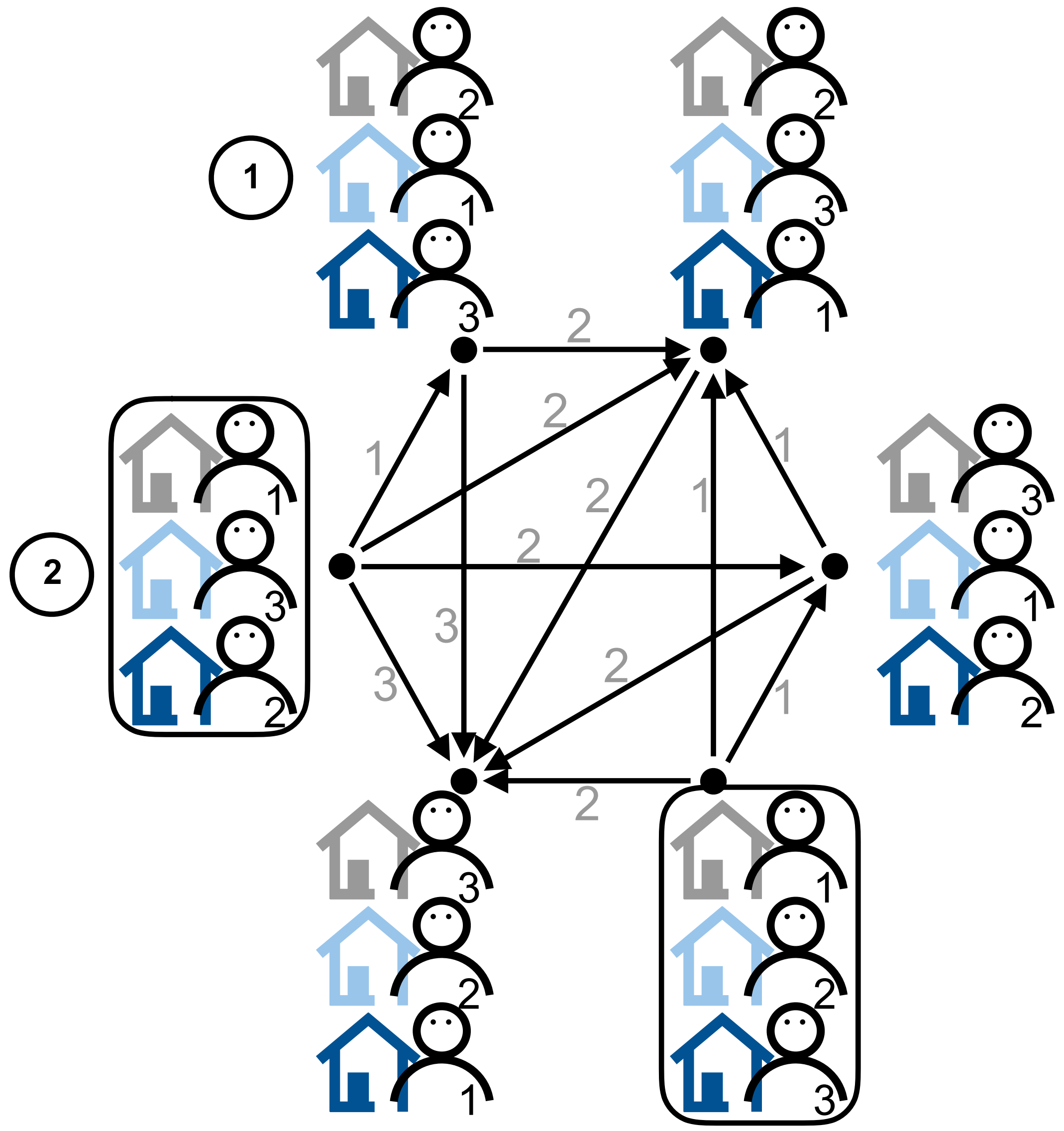


- **2** is **more popular** than **1**
- An assignment is **popular** if no more popular assignment exists (Gärdenfors, 1975)

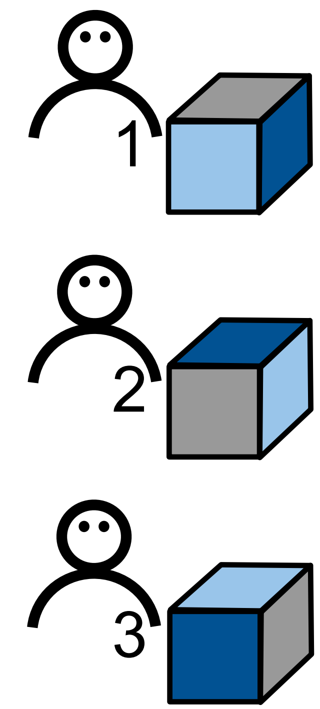
How to find popular assignments: the majority graph

- Directed (weighted) graph *G*
  - one vertex per assignment
  - majority margin as edge weights

- Popular assignments are **weak Condorcet winners** in the majority graph
  - do not have to exist…
  - …but randomized versions exist



Popular random assignments



- **Random assignment**: probability distribution over assignments
- A random assignment is **popular** if no other random assignment is preferred by an expected majority of agents
  - existence guaranteed by Minimax Theorem (Kavitha et al., 2011)
- Computation only requires majority graph
  - Which assignment problems induce identical majority graphs?
- Set of popular random assignments is convex
  - When is there a unique popular random assignment?
- Popularity is incompatible with strong envy-freeness and strong strategyproofness (Aziz et al., 2013)
  - What about weak envy-freeness and weak strategyproofness?

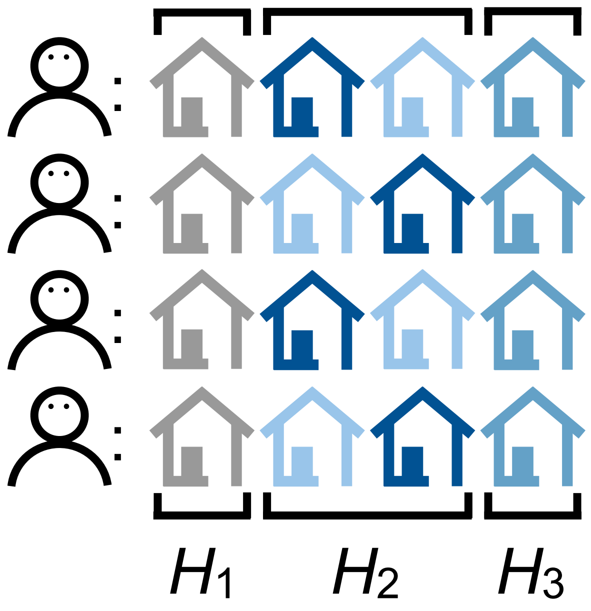
Identical majority graphs: decomposition

- **Decomposition**: Partition houses H into maximal number of subsets $H_1, \ldots, H_k$ s.t.

$$H_i > H_j \Leftrightarrow i < j$$

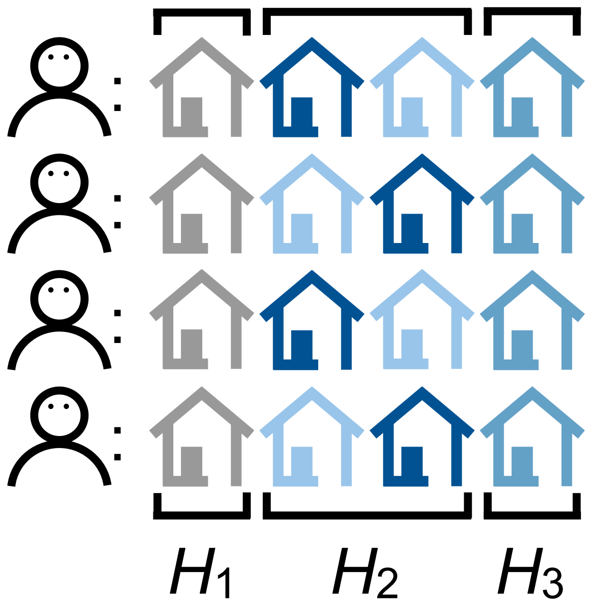
- Two decompositions are **rotation equivalent** if one can be obtained from the other by rotation of $H_1, \ldots, H_k$

- **Theorem**: Two assignment problems induce identical majority graphs iff their decompositions are rotation equivalent



Identical majority graphs: decomposition

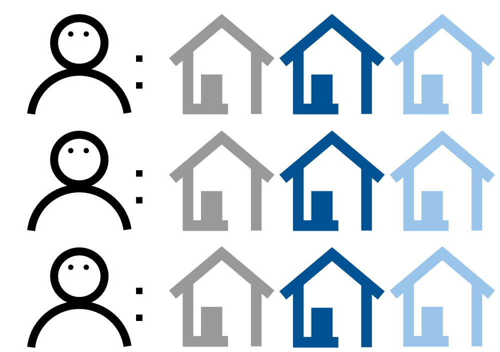
- **Theorem**: Two assignment problems induce identical majority graphs iff their decompositions are rotation equivalent



- Check in poly. time if two assignment problems induce identical majority graphs
- Check in poly. time if given majority graph is induced by some assignment problem
- Rotation equivalent decompositions imply identical popular random assignments
- Vast majority of profiles induce unique majority graph

Uniqueness of popular random assignments

- Assumption: agents share identical preferences



- **Theorem**: A random assignment p is popular iff $p_{i,j} = p_{i,j+2}$ for all i, j

- Odd n: unique popular random assignment: $p_{i,j} = {}^1/_n$

$$p = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

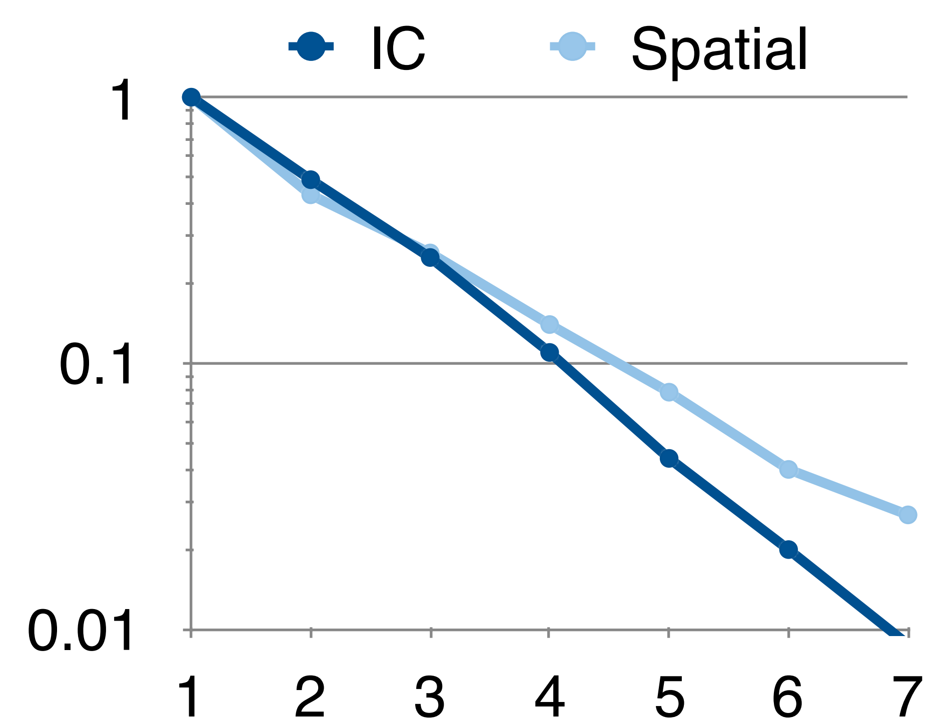
- Even n: infinitely many popular random assignments, for $n = 4$ e.g. q and q'

$$q = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix} \qquad q' = \begin{pmatrix} 1/3 & 1/6 & 1/3 & 1/6 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

Uniqueness of popular random assignments

- No obvious criterion for uniqueness

- Explicit computation infeasible for larger *n*
  - ~10^{17} profiles for $n = 6$
- Computer experiments to gain an insight
  - preferences sampled by Impartial Culture or Spatial (2-dim euclidean)
  - 10 000 samples for each *n*



- Fraction of assignment problems admitting unique popular random assignment decreases exponentially in *n*

Popularity vs. envy-freeness

- **Strongly envy-free**: own assignment preferred to all other for all vNM fct.'s
  - violated if someone's assignment preferred according to **some** vNM fct.

- **Weakly envy-free**: no one's assignment preferred to own for all vNM fct.'s
  - violated if someone's assignment preferred according to **all** vNM fct.'s

- Theorem (Aziz et al., 2013): Popularity and strong envy-freeness are incompatible for some assignment problem ($n \geq 3$)

- **Theorem**: Popularity and weak envy-freeness are incompatible for some assignment problem ($n \geq 5$)

Popularity vs. strategyproofness

- **Strongly strategyproof**: Truth-telling is preferred to lying for all vNM fct.'s
  - violated if manipulation possible for **some** vNM fct.

- **Weakly strategyproof**: Lying is never preferred to truth-telling (for all vNM fct.'s)
  - violated if manipulation possible for **all** vNM fct.'s

- Theorem (Aziz et al., 2013): Popularity and strong strategyproofness are incompatible ($n \geq 3$)

- **Theorem**: Popularity and weak strategyproofness are incompatible ($n \geq 7$)

Conclusion

- **Main contributions**
  - equivalence theorem linking assignment problems and majority graphs
  - analysis of uniqueness of popular random assignments
  - solution to two open problems by Aziz et al. (2013) regarding weak envy-freeness and weak strategyproofness

- **Open problem**
  - Does the impossibility for weak strategyproofness also hold for efficiency w.r.t. *pairwise comparison* instead of popularity?