Universal Pareto Dominance and Welfare for Plausible Utility Functions

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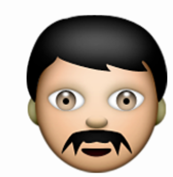
Problem and Motivation

Let *a*, *b*, and *c* be from an abstract set of alternatives (e.g., allocations, matchings, or coalition structures).

strongly prefers *a* and *b* to *c*, her preference between *a* and *b* is unknown.



strongly prefers *a* and *c* to *b*, his preference between *a* and *c* is unknown.

- How can we verify whether a given lottery is efficient?
- How can we find an efficient lottery?
- Is the convex combination of two efficient lotteries efficient?
- Does efficiency only depend on a lottery's support?

Outline

- Skew-symmetric bilinear utility and von Neumann-Morgenstern utility
- Generalization of the efficiency welfare theorem
  - Equivalence of Pareto efficiency and welfare maximization
  - Efficiency welfare theorems: McLennan, 2002 (ordinal efficiency); Manea, 2008 (constructive proof); Carroll, 2010 (plausible *vNM* utility functions)
- Ordinal efficiency for plausible utility functions
  - New notions of ordinal efficiency
  - Geometric and computational properties of sets of efficient lotteries

Preliminaries

- $N = \{1,\dots,n\}$ a finite set of agents.
- $A = \{a,b,c,\dots\}$ a finite set of alternatives.
  - $\Delta(A)$ the set of lotteries over A, e.g., $\frac{2}{3}a + \frac{1}{3}b$
- ϕ_i the set of plausible *SSB* utility functions of agent i.
  - $\phi = \phi_1 \times \dots \times \phi_n$ a (set-valued) utility profile
- $R_i \subseteq A \times A$ the preference relation of agent i.
  - P_i and I_i denote the strict and indifference part of R_i, respectively

vNM Utility and *SSB* Utility

- *vNM* utility (von Neumann & Morgenstern, 1947): linear function u_i: $\Delta A \to \mathbb{R}$; $u_i(p) = p^T u_i$.
  - violations of the independence and transitivity axiom in real world examples, e.g., preference reversal phenomenon (Grether and Plott, 1979)

$$u_i = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

- *SSB* utility (Fishburn, 1988): skew-symmetric and bilinear function φ_i: $\Delta A \times \Delta A \to \mathbb{R}$; $\varphi_i(p,q) = p^T \varphi_i\, q$.
  - *SSB* utility theory generalizes *vNM* utility theory

$$\varphi_i = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

Universal Pareto Undominatedness and Welfare Maximization

- $p\ R^{\varphi}\ q$ if $\varphi_i(p,q) \geq 0$ for all $i \in N$, where $\varphi = (\varphi_1,\dots,\varphi_n)$.
- A lottery p is Pareto efficient w.r.t. φ if there is no q such that $q\ P^{\varphi}\ p$.
- A lottery p is **universally (Pareto) undominated** w.r.t. Φ if there is no q such that $q\ P^{\varphi}\ p$ for all $\varphi \in \Phi$.
- A lottery p is affine welfare maximizing for φ if there is $\lambda \in \mathbb{R}_+^{|A|}$ such that $\lambda^T\varphi(p,q) \geq 0$ for all q.

Efficiency Welfare Theorem

Main theorem:

Let $\Phi_1,\ldots,\Phi_n$ be non-empty, convex, and relatively open sets of *SSB* utility functions and $\Phi = \Phi_1 \times \ldots \times \Phi_n$.

Then p is universally Pareto undominated w.r.t. Φ iff p is affine welfare maximizing for some $\varphi \in \Phi$.

Example: $A = \{a,b,c\}$, $p = \frac{2}{3}a + \frac{1}{3}b$,
$\Phi_1 = \{\varphi\colon \varphi(a,c) = \varphi(b,c) = 1,\ \varphi(a,b) \in (-1,1)\}$,
$\Phi_2 = \{\varphi\colon \varphi(a,b) = \varphi(c,b) = 1,\ \varphi(a,c) \in (-1,1)\}$.

$$\varphi = 2/3 \begin{pmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} + 1/3 \begin{pmatrix} 0 & 1 & 1/2 \\ -1 & 0 & -1 \\ -1/2 & 1 & 0 \end{pmatrix}$$

Universal Undominatedness and Ordinal Efficiency

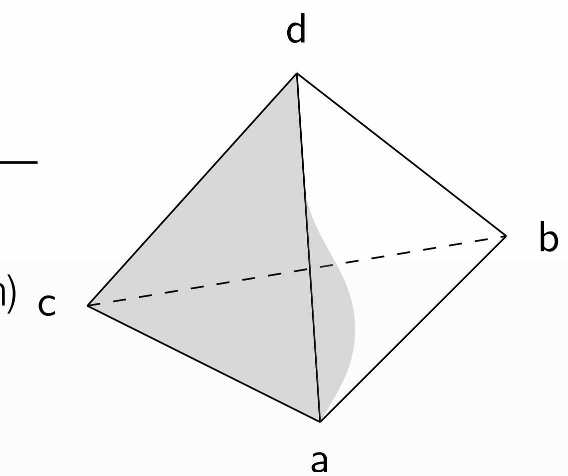
φ_i is consistent with R_i if $\varphi_i(a,b) \geq 0$ iff $a \; R_i \; b$ for all $a, b \in A$.

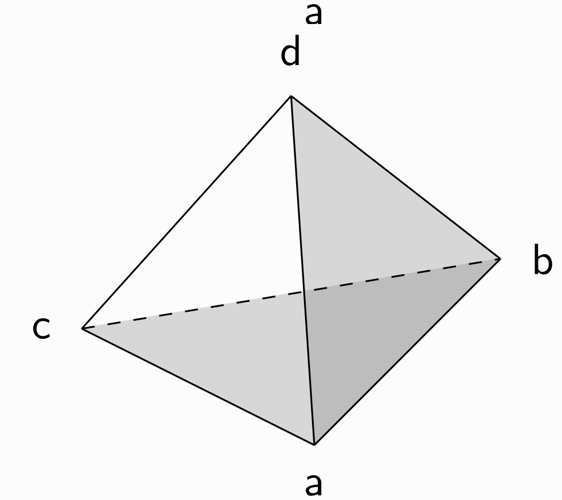
- *BD*-efficiency (bilinear dominance): universal undominatedness where $\phi_i^{SSB} = \{$all *SSB* utility functions consistent with $R_i\}$.
- *SD*-efficiency (stochastic dominance): universal undominatedness where $\phi_i^{vNM} = \{$all *vNM* utility functions consistent with $R_i\}$.
- *PC*-efficiency (pairwise comparison): universal undominatedness where $\phi_i^{PC} = \{\varphi_i^{PC}\}$, where $\varphi_i^{PC}(a,b) = \begin{cases} 1 & \text{if} \\ 0 & \text{if} \\ -1 & \text{if} \end{cases}$

$$PC\text{-efficiency} \Rightarrow SD\text{-efficiency} \Rightarrow BD\text{-efficiency}$$

Properties of the Set of Efficient Lotteries

Efficiency notion	*BD*	*ex post*	*SD*	*PC*
Existence	✔	✔	✔	✔ (minimax theorem)
Support dependence	✔	✔	✔	✗
Convexity	✔	✔	✗	✗
Finding and verification via	linear time algorithm	linear time algorithm	LP	LP

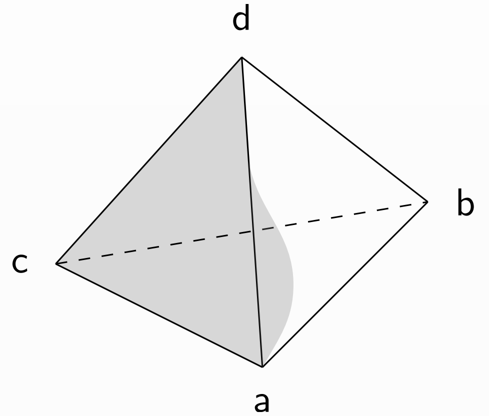


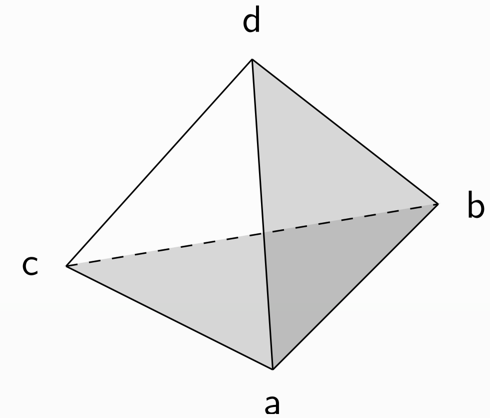


Summary

universal Pareto undominatedness ⇔ affine welfare maximization

Efficiency notion	*BD*	*ex post*	*SD*	*PC*
Existence	✔	✔	✔	✔ (minimax theorem)
Support dependence	✔	✔	✔	✘
Convexity	✔	✔	✘	✘
Finding and verification via	linear time algorithm	linear time algorithm	LP	LP





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