

Computational Social Choice*

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1 Introduction

Social choice theory concerns the design and formal analysis of methods for aggregating the preferences of multiple agents. Examples of such methods include voting procedures, which are used to aggregate the preferences of voters over a set of candidates standing for election to determine which candidate should win the election (or, more generally, to choose an alternative from a set of alternatives), or protocols for deciding on a fair allocation of resources given the preferences of a group of stake-holders over the range of bundles they might receive. Originating in economics and political science, social choice theory has since found its place as one of the fundamental tools for the study of multiagent systems. The reasons for this development are clear: if we view a multiagent system as a “society” of autonomous software agents, each of which has different objectives, is endowed with different capabilities, and possesses different information, then we require clearly defined and well-understood mechanisms for aggregating their views so as to be able to make collective decisions in such a multiagent system.

Computational social choice, the subject of this chapter, adds an algorithmic perspective to the formal approach of social choice theory. More broadly speaking, computational social choice deals with the application of methods usually associated with computer science to problems of social choice.

1.1 Introductory Example

Let us begin with a simple example. We shall discuss it at length, in order to introduce some of the key concepts that will be treated more formally later in the chapter. Consider the following situation in which there are four Dutchmen, three Germans, and two Frenchmen who have to decide which drink will be served for lunch (only a single drink will be served to all).¹ The Dutchmen prefer Milk to Wine to Beer, the Germans prefer Beer to Wine to Milk, and the Frenchmen prefer Wine to Beer to Milk. These preferences can be conveniently represented in a table where each group of agents is represented by one column.

¹This is based on an example used by Donald G. Saari at a conference in Rotterdam, where only Milk was served for lunch.

4	3	2
Milk	Beer	Wine
Wine	Wine	Beer
Beer	Milk	Milk

Now, which drink should be served based on these individual preferences? Milk could be chosen on the grounds that it has the most agents ranking it first (the Dutch). That is, it is the winner according to the *plurality rule*, which only considers how often each alternative is ranked in first place. However, a majority of agents (the Germans and the French) will be dissatisfied with this choice as they prefer *any* other drink to Milk. In fact, it turns out that Wine is preferred to both Beer and Milk by a 6:3 and a 5:4 majority of voters, respectively. An alternative with this property (defeating every other alternative in pairwise majority comparisons) is called a *Condorcet winner*. Yet another method of determining a collective choice would be to successively eliminate those beverages that are ranked first by the lowest number of agents (known as *Single Transferable Vote*, or STV).

This would result in Wine being eliminated first because only two agents (the French) rank it first. Between the remaining two options, Beer is ranked higher by the Germans and the French, and will eventually be chosen. In summary, this example shows that collective choice is not a trivial matter, as different, seemingly reasonable, voting rules can yield very different results.

Another important lesson that can be learned from this example concerns *strategic manipulation*. Assume the collective choice is determined using the plurality rule. Since preferences are private and each agent only knows his own preferences with certainty, nobody can prevent the Germans from *claiming* that their most-preferred drink is Wine. This will result in a more preferable outcome to them than reporting their preferences truthfully, because they get Wine rather than Milk, their least-preferred alternative. A seminal result in social choice theory, the Gibbard-Satterthwaite Theorem (discussed in detail in Section 3.2.1), states that *every* reasonable voting rule is susceptible to this type of manipulation.

While the example was carefully set up to avoid this, plurality and STV, as well as many other rules, can, in general, result in multiple alternatives ending up tied. If a social decision must be made, then we need to break this tie in some way—for example, by flipping a coin (resulting in a randomized rule), lexicographically according to the names of the alternatives, or by using another voting rule as a tie-breaking rule (whose own ties may yet again need to be broken). Another option is simply to “pass the buck” and declare all the tied alternatives

to be winners, so that the output is now a subset of the alternatives. For obvious reasons, we will generally require this subset to be nonempty. This sounds trivial, but, for example, as we will see later in the chapter, a given election may not have a Condorcet winner at all. As a consequence, the Condorcet winner method is not even a well-defined voting rule. Many voting rules, however, are so-called *Condorcet extensions*, which means that they choose the Condorcet winner whenever one exists. This is sometimes also called the Condorcet principle. Our example above shows that neither plurality nor STV are Condorcet extensions. An example of a rule that is a Condorcet extension is *Copeland's rule*, which chooses those alternatives that win the most pairwise majority comparisons. If no Condorcet winner exists, Copeland's rule still yields one or more winners. A disadvantage of Copeland's rule, in contrast to, say, the plurality rule when applied to elections with many voters and few alternatives, is that ties seem more likely here. In general, we prefer to end up with as small a set of winners as possible, but as we will see, this needs to be traded off against other properties.

We may even be a bit more ambitious and attempt to not only choose the winner(s), but rather to rank all the alternatives, representing "society's preferences" over them. This can be useful, for example, if we are worried that some alternatives may turn out to be unavailable and we need to quickly switch to another one. We may also be interested in the aggregate ranking for other reasons; for example, consider the problem of running a single query on multiple Internet search engines, and trying to aggregate the results into a single ranking. In the example above, we can simply rank the alternatives according to their pairwise majority comparisons: Wine defeats both Beer and Milk in their pairwise comparisons and so should be ranked first, and Beer defeats Milk and so should be ranked second. As we will see later, however, this approach can result in cycles. A simple approach to ranking alternatives is to use a rule that gives each alternative a score—such as plurality or Copeland—and sort the alternatives by aggregate score. (Note that if the pairwise majority approach does not result in cycles, then Copeland will agree with it.) For STV, one possibility is to sort the alternatives in inverse order of elimination. A generally applicable approach is to take the winners, rank them first, then vote again over the remaining alternatives, and to continue in this fashion until all alternatives have been ranked.

We shall revisit several of these ideas again later on, when we define the frameworks for social choice outlined here in more formal detail.

1.2 History of the Field

There are a number of historical cases showing that the intricacies of social choice have occupied people’s minds for a very long time [160]. Examples include the writings of Pliny the Younger, a senator in ancient Rome around the turn of the 1st century A.D.; the 13th century Catalan philosopher, alchemist, and missionary Ramon Llull; and the Marquis de Condorcet, a public intellectual who was active around the time of the French Revolution.

Social choice theory *as a scientific discipline* with sound mathematical foundations came into existence with the publication of the Ph.D. thesis of Kenneth J. Arrow in 1951 [5], who introduced the axiomatic method into the study of aggregation methods and whose seminal *Impossibility Theorem* shows that any such method that satisfies a list of seemingly basic fairness requirements must in fact amount to a dictatorial rule. Since then, much of the work in classical social choice theory has focused on results concerning the formal possibility and impossibility of aggregation methods that combine certain desirable properties—like *Pareto-optimality*, *monotonicity*, or *non-manipulability*—without resulting in an unacceptable concentration of power. Some of the landmark results include Sen’s characterization of preference domains allowing for consistent majority decisions [197] and the Gibbard-Satterthwaite Theorem [124, 191] mentioned earlier, which establishes the impossibility of devising a reasonable, general voting rule that is immune to strategic manipulation.

The first clear examples of work in *computational* social choice are a series of papers by Bartholdi, Orlin, Tovey, and Trick, published around 1990 [19, 20, 17]. They argued that complexity theory, as studied in theoretical computer science, is relevant to social choice. For instance, they analyzed the complexity of determining the winners in an intricate voting rule due to C.L. Dodgson, better known as Lewis Carroll, the author of “Alice in Wonderland”.² They also fielded the fundamental idea that complexity barriers might provide a means of protection against strategic manipulation and other undesirable behavior. That is, while classical social choice theory showed that it is a mathematical impossibility to devise a voting rule that cannot be manipulated, computer science might provide the tools for making this unwanted behavior so difficult that it can be neglected in practice.³

²It was shown later that determining the winners according to Dodgson’s rule is complete for the complexity class Θ_2^P [130]. This is remarkable as Θ_2^P was considered to lack “natural” complete problems and Dodgson’s rule was proposed long before complexity theory existed.

³As we shall see, this approach of using computational complexity as a barrier against strategic manipulation has its limitations, but conceptually this has nevertheless been an important idea

This groundbreaking work was followed by a small number of isolated publications throughout the 1990s. In the first few years of the 21st century, as the relevance of social choice to artificial intelligence, multiagent systems, and electronic commerce became apparent, the frequency of contributions on problems related to social choice with a computational flavor suddenly intensified. Although the field was still lacking a name, by 2005 contributions in what we would now call “computational social choice” had become a regular feature at several of the major conferences in artificial intelligence. The first workshop specifically dedicated to computational social choice, and the first event to explicitly use this name, took place in 2006 [102]. Around the same time, Chevaleyre et al. [56] attempted the first classification of research in the area by distinguishing (a) the nature of the social choice problem addressed, and (b) the type of formal or computational technique used.

1.3 Applications

Social choice theory was originally developed as an abstraction of problems that arise in political science and economics. More generally, social choice theory provides a useful theoretical framework for the precise mathematical study of the normative foundations of collective decision making, in a wide range of areas, involving not only human decision-makers but also autonomous software agents. This chapter will focus on the theoretical foundations of computational social choice. But before we delve into the theory, let us briefly cite a few examples of actual and potential application domains, going beyond political elections and collective decision making in multiagent systems, where the methods we shall cover in this chapter can be put to good use.

The first such example comes from the domain of Internet search engines. Imagine you want to design a *meta search engine* that combines the search results of several engines. This problem has a lot in common with preference aggregation. Aggregating preferences means asking each individual agent for a ranking over the set of alternatives and then amalgamating this information into a single such ranking that adequately represents the preferences of the group. For the meta search engine, we ask each individual search engine for a ranking of its own, say, 20 top results and then have to aggregate this information to produce our meta ranking. Of course, the problems are not exactly the same. For instance, some website may not have been ranked at all by one search engine, but be in the top 5

which has inspired a good deal of exciting research.

for another. Also, the general principles that we might want to adhere to when performing the aggregation might differ: in preference aggregation, fairness will play an important role; when aggregating search results fairness is not a goal in itself. Nevertheless, it is clear that insights from social choice theory can inform possible approaches for designing our meta search engine. In fact, this situation is rather typical in computational social choice: for many modern applications, we can rely on some of the basic insights from social choice theory, but to actually develop an adequate solution, we do have to alter some of the classical assumptions.

There is also a less obvious application of principles of social choice to search engines. One way of measuring the *importance* of a webpage is the number of other webpages linking to it. In fact, this is a recursive notion: the importance of our webpage also depends on the importance of the pages linking to it, which in turn depends on the importance of the pages linking to those. This idea is the basis for the PAGERANK algorithm at the core of Google's search engine [170]. We may think of this as an election where the set of the voters and the set of the candidates coincide (both are the set of all webpages). In this sense, the ranking of the importance of webpages may be considered as a social choice problem. This perspective has led to a deeper understanding of the problem, for instance, by providing an axiomatic characterization of different ranking algorithms [3].

Another example of an application domain for which the perspective of social choice theory can provide fruitful new insights is that of *recommender systems*. A recommender system is a tool for helping users choose attractive products on the basis of choices made by other users in the past. An important technique in this field is *collaborative filtering*. By reinterpreting collaborative filtering as a process of preference aggregation, the axiomatic method developed in social choice theory has proven helpful in assessing and comparing the quality of different collaborative filtering approaches [171].

Yet another example is the problem of *ontology merging*, which arises in the context of the *Semantic Web*. Suppose different information providers on the Semantic Web provide us with different ontologies describing the same set of concepts. We would like to combine this information so as to arrive at the best possible ontology representing the available knowledge regarding the problem domain. This is a difficult problem that will require a combination of different techniques. Social choice theory can make a contribution in those cases where we have little information regarding the reliability of the individual providers and can only resort to aggregating whatever information they provide in a "fair" (and logically consistent) manner [174].

We shall allude to further areas of application along the way. However, our

focus will be on theoretical foundations from here on.

1.4 Chapter Outline

In this chapter we review the foundations of social choice theory and introduce the main research topics in computational social choice that have been identified to date. Specifically, Section 2 introduces the axiomatic framework for studying preference aggregation and discusses the most important seminal result in the field, Arrow's Theorem, in detail. Section 3 is an introduction to voting theory. We present the most important voting rules and then focus on the problem of strategic manipulation. This includes a discussion of the Gibbard-Satterthwaite Theorem and a number of possible avenues for circumventing the impossibility it is pointing to. Section 4 focuses on voting scenarios where the set of alternatives to choose from has a combinatorial structure, as is the case when we have to elect a committee (rather than a single official) or more generally when we have to collectively decide on an instantiation of several variables. In Section 5 we turn our attention to the problem of fairly allocating a number of goods to a group of agents and discuss the problems that are characteristic for this particular type of social choice problem. Section 6 concludes with a brief discussion of related research topics in computational social choice not covered in this chapter and with a number of recommendations for further reading.

2 Preference Aggregation

One of the most elementary questions in social choice theory is how the preference relations of individual agents over some abstract set of alternatives can be aggregated into one collective preference relation. Apart from voting, this question is of broad interest in the social sciences, because it studies whether and how a society of autonomous agents can be treated as a single rational decision-maker. As we point out in Section 2.1, results in this framework are very discouraging.

In many practical settings, however, one is merely interested in a set of socially acceptable alternatives rather than a collective preference relation. In Section 2.2, we discuss the relationship between both settings and present some positive results for the latter framework.

2.1 Social Welfare Functions

We start by investigating social welfare functions, the simplest and perhaps most elegant framework of preference aggregation. A social welfare function aggregates preferences of individual agents into collective preferences. More formally, we consider a finite set $N = \{1, \dots, n\}$ of at least two agents (sometimes also called individuals or voters) and a finite universe U of at least two alternatives (sometimes also called candidates). Each agent i entertains preferences over the alternatives in U , which are represented by a transitive and complete preference relation \succeq_i . *Transitivity* requires that $a \succeq_i b$ and $b \succeq_i c$ imply $a \succeq_i c$ for all $a, b, c \in U$, and *completeness* requires that any pair of alternatives $a, b \in U$ is comparable, i.e., it holds that either $a \succeq_i b$ or $b \succeq_i a$ or both. In some cases, we will assume preferences to be linear, i.e., also satisfying antisymmetry ($a \succeq_i b$ and $b \succeq_i a$ imply that $a = b$), but otherwise we impose no restrictions on preference relations. We have $a \succeq_i b$ denote that agent i likes alternative a at least as much as alternative b and write \succ_i for the strict part of \succeq_i , i.e., $a \succ_i b$ if $a \succeq_i b$ but not $b \succeq_i a$. Similarly, \sim_i denotes i 's indifference relation, i.e., $a \sim_i b$ if both $a \succeq_i b$ and $b \succeq_i a$. The set of all preference relations over the universal set of alternatives U will be denoted by $\mathcal{R}(U)$. The set of *preference profiles*, associating one preference relation with each individual agent, is then given by $\mathcal{R}(U)^n$.

Economists often also consider *cardinal* (rather than *ordinal*) preferences, which are usually given in the form of a utility function that assign numerical values to each alternative. It is easy to show that, for a finite number of alternatives, a preference relation can be represented by a utility function if and only if it satisfies transitivity and completeness (see Exercise 1). Still, a utility function may yield much more information than a preference relation, such as the intensity of preferences, as well as preferences over probability distributions over the alternatives. In the absence of a common numeraire such as money, the meaning of individual utility values and especially the inter-personal comparisons between those values is quite controversial. Therefore, the ordinal model based on preference relations is the predominant model in abstract social choice theory. In special domains such as fair division (see Section 5), however, cardinal preferences are also used.

A social welfare function is a function that maps individual preference relations to a collective preference relation.

Definition 1. A social welfare function (SWF) is a function $f : \mathcal{R}(U)^n \rightarrow \mathcal{R}(U)$.

For a given preference profile $R = (\succeq_1, \dots, \succeq_n)$, the resulting *social* preference relation will be denoted by \succeq .

It was the Marquis de Condorcet who first noted that the concept of a social preference relation can be problematic. When there are just two alternatives, common sense and several axiomatic characterizations, such as May’s Theorem [157], suggest that alternative a should be socially preferred to alternative b if and only if there are more voters who strictly prefer a to b than b to a . This concept is known as *majority rule*. Since the majority rule provides us with social pairwise comparisons, it appears to be a natural candidate for an SWF. However, as demonstrated by the *Condorcet paradox* [86], the majority rule can result in cycles when there are more than two alternatives. To see this, consider the preference relations of three voters given in Figure 1. A majority of voters (two out of three) prefers a to b . Another majority prefers b to c and yet another one c to a . Clearly, the pairwise majority relation in this example is cyclic and therefore not a well-formed preference relation. Hence, the majority rule does not constitute an SWF.

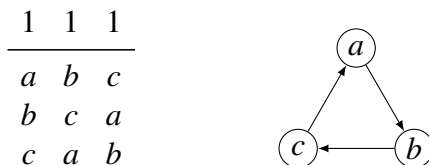


Figure 1: Condorcet’s paradox [86]. The left-hand side shows the individual preferences of three agents such that the pairwise majority relation, depicted on the right-hand side, is cyclic.

In what is perhaps the most influential result in social choice theory, Arrow [5] has shown that this “difficulty in the concept of social welfare” (as he calls it) is not specific to the majority rule, but rather applies to a very large class of SWFs. Arrow’s Impossibility Theorem states that a seemingly innocuous set of desiderata cannot be simultaneously met when aggregating preferences. These desiderata are *Pareto-optimality*, *independence of irrelevant alternatives*, and *non-dictatorship*; they are defined as follows.

- An SWF satisfies *Pareto-optimality* if strict unanimous agreement is reflected in the social preference relation. Formally, Pareto-optimality requires that $a \succ_i b$ for all $i \in N$ implies that $a \succ b$.
- An SWF satisfies *independence of irrelevant alternatives (IIA)* if the social preference between any pair of alternatives only depends on the individual preferences restricted to these two alternatives. Formally, let R and R' be two preference profiles and a and b be two alternatives such that $R|_{\{a,b\}} =$

$R'_{\{a,b\}}$, i.e., the pairwise comparisons between a and b are identical in both profiles. Then, IIA requires that a and b are also ranked identically in \succsim , i.e., $\succsim_{\{a,b\}} = \succsim'_{\{a,b\}}$.

- An SWF is *non-dictatorial* if there is no agent who can dictate a strict ranking no matter which preferences the other agents have. Formally, an SWF is non-dictatorial if there is no agent i such that for all preference profiles R and alternatives a, b , $a \succ_i b$ implies that $a \succ b$.

Theorem 1 (Arrow, 1951). *There exists no SWF that simultaneously satisfies IIA, Pareto-optimality, and non-dictatorship whenever $|U| \geq 3$.*

According to Paul Samuelson, who is often considered the founding father of modern economics, Arrow’s Theorem is one of the significant intellectual achievements of the 20th century [188]. A positive aspect of such a negative result is that it provides boundaries on what can actually be achieved when aggregating preferences. In particular, Arrow’s Theorem shows that at least one of the required conditions has to be omitted or relaxed in order to obtain a positive result. For instance, if $|U| = 2$, IIA is trivially satisfied by *any* SWF and reasonable SWFs (such as the majority rule) also satisfy the remaining conditions. In a much more elaborate attempt to circumvent Arrow’s Theorem, Young [231] proposed to replace IIA with *local IIA (LIIA)*, which only requires IIA to hold for consecutive pairs of alternatives in the social ranking. By throwing in a couple of other conditions (such as anonymity and neutrality, which will be defined in Section 3) and restricting attention to linear individual preferences, Young completely characterizes an aggregation function known as *Kemeny’s rule*.

Kemeny’s Rule. Kemeny’s rule [140] yields all strict rankings that agree with as many pairwise preferences of the agents as possible. That is, it returns

$$\arg \max_{\succ} \sum_{i \in N} |\succ \cap \succ_i|.$$

Since there can be more than one ranking that satisfies this property, Kemeny’s rule is not really an SWF but rather a multi-valued SWF. (Young refers to these as *social preference functions*.) Alternatively, Kemeny’s rule can be characterized using maximum likelihood estimation [231, 232].⁴ Over the years, Kemeny’s rule

⁴This is done under a model where there exists a “correct” ranking of the alternatives, and the agents’ preferences are noisy estimates of this correct ranking. This result relies on a particular

has been reinvented by many scholars in different fields. It is also known as the *median* or *linear ordering* procedure [15, 53]. Kemeny’s rule is not only very interesting from an axiomatic but also from a computational point of view. The problem of computing a Kemeny ranking, as well as the closely related problem of computing a *Slater* ranking (a ranking that agrees with the outcomes of as many pairwise elections as possible), correspond to a computational problem on graphs known as the *minimum feedback arc set problem* (in the case of Kemeny’s rule, the weighted version of this problem). It has been shown that computing a Kemeny ranking is NP-hard [20], even when there are just four voters [97]. Moreover, deciding whether a given alternative is ranked first in a Kemeny ranking is Θ_2^P -complete [131]. Nevertheless, under certain conditions, there is a polynomial-time approximation scheme (PTAS) for the Kemeny problem [141]. For further details on these problems, we refer to the works of Davenport and Kalagnanam [84], Conitzer et al. [73], Alon [2], Conitzer [61], Charon and Hudry [53], Betzler et al. [23], Hudry [135], Betzler et al. [26], Brandt et al. [48], Hudry [136], and Ali and Meila [1].⁵

Rather than relaxing the explicit conditions in Arrow’s Theorem, one may call its implicit assumptions into question. For instance, in many applications, a full social preference relation is not needed; rather, we just wish to identify the socially most desirable alternatives. This corresponds to the framework considered in the following section.⁶

2.2 Social Choice Functions

The central objects of study in this section are *social choice functions*, i.e., functions that map the individual preferences of the agents and a feasible subset of the alternatives to a set of socially preferred alternatives, the choice set. Through-

noise model; if the noise model is changed, the maximum likelihood solution can result in other SWFs, though for yet other SWFs, it can be proved that no noise model would yield that SWF as the solution [90, 70, 209, 76]. However, the Kemeny result is robust to some other generalizations of the model [222, 66].

⁵In 1995, Peyton Young predicted “that the time will come when [Kemeny’s rule] is considered a standard tool for political and group decision making” [232]. This has not yet happened, but the website www.votefair.org provides an interface to use Kemeny’s rule for surveys, polls, and elections at no charge.

⁶This effectively reduces the codomain of the aggregation function. As we will see in Section 3.2.2, a common technique to avoid negative results in social choice theory is to reduce the *domain* of the function.

out this chapter, the set of possible feasible sets $\mathcal{F}(U)$ is defined as the set of all non-empty subsets of U . A feasible set (or agenda) defines the set of possible alternatives in a specific choice situation at hand. The reason for allowing the feasible set to vary is that we will later define properties that relate choices from different feasible sets to each other [see also 200, 206].

Definition 2. A social choice function (SCF) is a function $f : \mathcal{R}(U)^n \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ such that $f(R, A) \subseteq A$ for all R and A .

2.2.1 The Weak Axiom of Revealed Preference

Arrow's Theorem can be reformulated for SCFs by appropriately redefining Pareto-optimality, IIA, and non-dictatorship and introducing a new property called the weak axiom of revealed preference, as follows.

Pareto-optimality now requires that an alternative should not be chosen if there exists another feasible alternative that *all* agents unanimously prefer to the former—more precisely, $a \notin f(R, A)$ if there exists some $b \in A$ such that $b \succ_i a$ for all $i \in N$. An SCF f is non-dictatorial if there is no agent i such that for all preference profiles R and alternatives a , $a \succ_i b$ for all $b \in A \setminus \{a\}$ implies $a \in f(R, A)$.⁷ Independence of irrelevant alternatives reflects the idea that choices from a set of feasible alternatives should not depend on preferences over alternatives that are infeasible, i.e., $f(R, A) = f(R', A)$ if $R|_A = R'|_A$. Interestingly, in the context of SCFs, IIA constitutes no more than a framework requirement for social choice and is not the critical assumption it used to be in the context of SWFs.

Finally, the *weak axiom of revealed preference (WARP)* demands that choice sets from feasible sets are strongly related to choice sets from feasible subsets. Let A and B be feasible sets such that $B \subseteq A$. WARP requires that the choice set of B consists precisely of those alternatives in B that are also chosen in A , whenever this set is non-empty. Formally, for all feasible sets A and B and preference profiles R ,

$$\text{if } B \subseteq A \text{ and } f(R, A) \cap B \neq \emptyset \text{ then } f(R, A) \cap B = f(R, B). \quad (\text{WARP})$$

We are now ready to state a variant of Arrow's Theorem for SCFs.

Theorem 2 (Arrow, 1951, 1959). *There exists no SCF that simultaneously satisfies IIA, Pareto-optimality, non-dictatorship, and WARP whenever $|U| \geq 3$.*

⁷Theorem 2 holds for an even weaker notion of non-dictatorship in which a dictator can enforce that $\{a\} = f(R, A)$.

As the Arrovian conditions—Pareto-optimality, IIA, non-dictatorship, and WARP—cannot be satisfied by any SCF, at least one of them needs to be excluded or relaxed to obtain positive results. Clearly, dropping non-dictatorship is unacceptable and, as already mentioned, IIA merely states that the SCF represents a reasonable model of preference aggregation [see, e.g., 193, 29]. Wilson [215] has shown that without Pareto-optimality only SCFs that are constant (i.e., completely unresponsive) or fully determined by the preferences of a single agent are possible. Moreover, it could be argued that not requiring Pareto-optimality runs counter to the very idea of *social* choice. Accordingly, the only remaining possibility is to relax WARP.

2.2.2 Contraction and Expansion Consistency

Building on earlier work by Sen [198], Bordes [28] factorized WARP into two separate conditions by splitting the equality in the consequence of the definition of WARP into two inclusions. The resulting conditions are known as *contraction* and *expansion*.

Contraction prescribes that an alternative that is chosen from some feasible set will also be chosen from all subsets in which it is contained. Formally, SCF f satisfies contraction if for all A, B and R ,

$$\text{if } B \subseteq A \text{ then } B \cap f(R, A) \subseteq f(R, B). \quad (\text{contraction})$$

The intuition behind *expansion* is that if alternative a is chosen from some set that contains another alternative b , then it will also be chosen in all supersets in which b is chosen. Formally, SCF f satisfies expansion if for all A, B and R ,

$$\text{if } B \subseteq A \text{ and } B \cap f(R, A) \neq \emptyset \text{ then } f(R, B) \subseteq B \cap f(R, A). \quad (\text{expansion})$$

One possibility to escape the haunting impossibility of social choice is to require only contraction or expansion but not both at the same time. It turns out that contraction and even substantially weakened versions of it give rise to impossibility results that retain Arrow's spirit [199]. As an example, consider the preference profile given in Figure 1. All of the voting rules mentioned in the introduction (plurality, STV, and Copeland) will yield a tie between all three alternatives. Hence, if any of these rules were to satisfy contraction, they would need to yield both available alternatives in every two-element subset of $\{a, b, c\}$. However, this is not the case for *any* of these subsets as each of them has a single winner according to all three rules (in fact, it is a 2:1 majority in each case, so this would be the case for almost any natural rule).

Expansion consistency conditions, on the other hand, are much less restrictive. In fact, a number of appealing SCFs can be characterized using weakenings of expansion and inclusion-minimality. Inclusion-minimality is quite natural in this context as one is typically interested in choice sets that are as small as possible.⁸ These characterizations are nice examples of how positive results can be obtained by using the axiomatic method.

In the following, an SCF is said to be *majoritarian* if choice sets only depend on the majority rule within the feasible set. For technical reasons, we furthermore assume that n is odd and that the preferences of the voters are strict, which guarantees that the majority rule is asymmetric.

Top Cycle. The top cycle is the smallest majoritarian SCF satisfying expansion [28]. It consists of the maximal elements of the transitive closure of the weak majority relation [88, 45] and can be computed in linear time by using standard algorithms for identifying strongly connected components in digraphs such as those due to Kosaraju or Tarjan [see, e.g., 80].

Uncovered Set. The uncovered set is the smallest majoritarian SCF satisfying a weak version of expansion [164]. Interestingly, the uncovered set consists precisely of those alternatives that reach every other alternative on a majority rule path of length at most two [202]. Based on this characterization, computing the uncovered set can be reduced to matrix multiplication and is thus feasible in almost linear time [134, 42].

Banks Set. The Banks set is the smallest majoritarian SCF satisfying a weakening of weak expansion called strong retentiveness [40]. In contrast to the previous two SCFs, the Banks set cannot be computed in polynomial time unless P equals NP. Deciding whether an alternative is contained in the Banks set is NP-complete [216, 47]. Interestingly, some alternatives (and thus subsets) of the Banks set can be found in linear time [133]. A very optimized (exponential-time) algorithm for computing the Banks set was recently proposed by Gaspers and Mnich [122].⁹

⁸Moreover, due to the inclusive character of expansion consistency conditions, they are easily satisfied by very indiscriminatory SCFs. For instance, the trivial SCF, which always yields all feasible alternatives, trivially satisfies expansion (and all of its weakenings).

⁹Another SCF, the tournament equilibrium set [194], was, for more than 20 years, conjectured to be the unique smallest majoritarian SCF satisfying retentiveness, a weakening of strong retentiveness. This was recently disproven by Brandt et al. [49]. Deciding whether an alternative is

Two other SCFs, namely the *minimal covering set* [95] and the *bipartisan set* [147], have been axiomatized using a variant of contraction, which is implied by WARP [43]. While the bipartisan set can be computed using a single linear program, the minimal covering set requires a slightly more sophisticated, yet polynomial-time, algorithm [42]. In addition to efficient computability, the minimal covering set and the bipartisan set satisfy a number of other desirable properties [151, 39] (see also Section 3.2.5).

3 Voting

In the previous section, we started our formal treatment of social choice and encountered some of the fundamental limitations that we face. The purpose of presenting these limitations at the outset is of course not to convince the reader that social choice is hopeless and we should give up on it; it is too important for that. (One is reminded of Churchill’s quote that “democracy is the worst form of government except for all those other forms that have been tried from time to time.”) Rather, it is intended to get the reader to think about social choice in a precise manner and to have realistic expectations for what follows. Now, we can move on to some more concrete procedures for making decisions based on the preferences of multiple agents.

3.1 Voting Rules

We begin by defining voting rules.

Definition 3. A voting rule is a function $f : \mathcal{R}(U)^n \rightarrow \mathcal{F}(U)$.

Of course, every SCF can also be seen as a voting rule. There are two reasons we distinguish SCFs from voting rules. First, from a technical perspective, the SCFs defined in the previous section were axiomatized using variable feasible sets in order to salvage some degree of collective rationality. Second, some of these SCFs (e.g., the top cycle) can hardly be considered voting rules because they are not discriminatory enough. Of course, the latter is merely a gradual distinction, but there have been attempts to formalize this [see, e.g., 10, 207, 117, 195]. When ignoring all conditions that relate choices from different feasible sets

contained in the tournament equilibrium set of a tournament is NP-hard [47]. This problem is not known to be in NP and may be significantly harder.

with each other, we have much more freedom in defining aggregation functions. For simplicity, we assume throughout this section that preferences are linear, i.e., there are no ties in individual preference relations.

An important property that is often required of voting rules in practice, called *resoluteness*, is that they should always yield a unique winner. Formally, a voting rule f is *resolute* if $|f(R)| = 1$ for all preference profiles R . Two natural symmetry conditions are anonymity and neutrality. *Anonymity* requires that the outcome of a voting rule is unaffected when agents are renamed (or more formally, when the individual relations within a preference profile are permuted). In a similar vein, *neutrality* requires that a voting rule is invariant under renaming alternatives.

Unfortunately, in general, anonymous and neutral voting rules cannot be single-valued. The simplest example concerns two agents and two alternatives, each of which is preferred by one of the voters. Clearly, a single alternative can only be chosen by breaking anonymity or neutrality.¹⁰

In the remainder of this section, we will define some of the most common voting rules.

3.1.1 Scoring Rules

A common objection to the plurality rule is that an alternative ought to get some credit for being ranked, say, in second place by a voter. Under a (positional) scoring rule, each time an alternative is ranked i th by some voter, it gets a particular score s_i . The scores of each alternative are then added and the alternatives with the highest cumulative score are selected. Formally, for a fixed number of alternatives m , we define a *score vector* as a vector $s = (s_1, \dots, s_m)$ in \mathbb{R}^m such that $s_1 \geq \dots \geq s_m$ and $s_1 > s_m$. Three well-known examples of scoring rules are *Borda's rule*, the *plurality rule*, and the *anti-plurality rule*.

Borda's rule. Under *Borda's rule* alternative a gets k points from voter i if i prefers a to k other alternatives, i.e., the score vector is $(|U| - 1, |U| - 2, \dots, 0)$. Borda's rule takes a special place within the class of scoring rules as it chooses those alternatives with the *highest average rank* in individual rankings. While Borda's rule is not a Condorcet extension, it is the *only* scoring rule that never gives a Condorcet winner the lowest accumulated score [204]. Another appealing axiomatic characterization of Borda's rule was given by Young [228].

¹⁰Moulin [163] has shown that anonymous, neutral, and resolute voting rules exist if and only if $|U|$ can be written as the sum of non-trivial divisors of n .

Plurality rule. The score vector for the plurality rule is $(1, 0, \dots, 0)$. Hence, the cumulative score of an alternative equals the number of voters by which it is ranked first.

Anti-plurality rule. The score vector for the anti-plurality rule (which is sometimes also called *veto*) is $(1, \dots, 1, 0)$. As a consequence, it chooses those alternatives that are least-preferred by the lowest number of voters.

Due to their simplicity, scoring rules are among the most-used voting rules in the real world. Moreover, there are various elegant characterizations of scoring rules. In Section 2, we introduced axioms that impose consistency restrictions on choice sets when the set of feasible alternatives varies. Alternatively, one can focus on changes in the set of voters. A very natural consistency property with respect to a variable electorate, often referred to as *reinforcement*, was suggested independently by Smith [204] and Young [228]. It states that all alternatives that are chosen simultaneously by two disjoint sets of voters (assuming that there is at least one alternative with this property) should be precisely the alternatives chosen by the union of both sets of voters. When also requiring anonymity, neutrality, and a mild technical condition, Smith [204] and Young [229] have shown that scoring rules are the *only* voting rules satisfying these properties simultaneously.

A voting procedure, popularized by Brams and Fishburn [36], that is closely related to scoring rules is *approval voting*. In approval voting, every voter can approve any number of alternatives and the alternatives with the highest number of approvals win. We deliberately called approval voting a voting procedure, because technically it is not really a voting rule (unless we impose severe restrictions on the domain of preferences by making them dichotomous). Various aspects of approval voting (including computational ones) are analyzed in a recent compendium by Laslier and Sanver [152].

3.1.2 Condorcet Extensions

As mentioned in Section 1, a Condorcet winner is an alternative that beats every other alternative in pairwise majority comparisons. We have already seen in the Condorcet Paradox that there are preference profiles that do not admit a Condorcet winner. However, whenever a Condorcet winner does exist, it obviously has to be unique. Many social choice theorists consider the existence of Condorcet winners to be of great significance and therefore call any voting rule that picks a Condorcet winner whenever it exists a *Condorcet extension*. For aficionados of Condorcet's

criterion, scoring rules present a major disappointment: every scoring rule fails to select the Condorcet winner for some preference profile [118]. This is shown by using one universal example given in Figure 2.

$\begin{array}{cccc} 6 & 3 & 4 & 4 \\ \hline a & c & b & b \\ b & a & a & c \\ c & b & c & a \end{array}$	$\begin{array}{l} a\text{'s score : } 6 + 7s_2 \\ b\text{'s score : } 8 + 6s_2 \\ c\text{'s score : } 3 + 4s_2 \end{array}$
---	---

Figure 2: Example due to Fishburn [118], which shows that no scoring rule is a Condorcet extension. Scores for the score vector $(1, s_2, 0)$ are given on the right-hand side.

It is easily verified that alternative a is a Condorcet winner as 9 out of 17 voters prefer a to b and 10 out of 17 voters prefer a to c . Now, consider an arbitrary scoring rule with score vector (s_1, s_2, s_3) . Due to the linearity of scores, we may assume without loss of generality that $s_1 = 1$ and that $s_3 = 0$. The resulting scores for each alternative are given in Figure 2. Since $s_2 \in [0, 1]$, the score of alternative b always exceeds that of alternatives a and c . In other words, b is the unique winner in any scoring rule, even though a is the Condorcet winner.

We now give some examples of rules that do satisfy Condorcet's criterion. This list is far from complete, but it already shows the wide variety of Condorcet extensions.

Copeland's rule. We have already mentioned *Copeland's rule*: an alternative gets a point for every pairwise majority win, and some fixed number of points between 0 and 1 (say, $1/2$) for every pairwise tie. The winners are the alternatives with the greatest number of points.

Maximin. Under the *maximin* rule, we consider the magnitude of pairwise election results (by how many voters one alternative was preferred to the other). We evaluate every alternative by its worst pairwise defeat by another alternative; the winners are those who lose by the lowest margin in their worst pairwise defeats. (If there are any alternatives that have no pairwise defeats, then they win.)

Dodgson's rule. *Dodgson's rule* yields all alternatives that can be made a Condorcet winner by interchanging as few adjacent alternatives in the individual

rankings as possible. Deciding whether an alternative is a Dodgson winner is Θ_2^p -complete and thus computationally intractable [20, 130]. Various computational properties of Dodgson’s rule such as approximability and fixed-parameter tractability have been studied [see, e.g., 158, 51, 24, 52]. Unfortunately, Dodgson’s rule violates various mild axioms that almost all other Condorcet extensions satisfy [see, e.g., 38].

Young’s rule. *Young’s rule* is based on removing voters in order to obtain a Condorcet winner. More precisely, it yields all alternatives that can be made a Condorcet winner by removing as few voters as possible. Deciding whether an alternative is a winner according to Young’s rule is Θ_2^p -complete [185].¹¹ Further computational results for Young’s rule were obtained by Caragiannis et al. [51], Betzler et al. [24]

Nanson’s rule. *Nanson’s rule* is a runoff method similar to STV as described in Section 1.1. In Nanson’s original definition, a series of Borda elections is held and all alternatives who at any stage have no more than the average Borda score are excluded unless all alternatives have identical Borda scores, in which case these candidates are declared the winners. There exist two variants of Nanson’s rule due to Fishburn and Schwartz, which exclude candidates with the *lowest* Borda score (also known as *Baldwin’s rule*) and candidates whose Borda score is *less than* the average score, respectively [167].

Ranked pairs. The *ranked pairs* rule generates a ranking of all alternatives (and the first-ranked alternative can be considered the winner). It first sorts all pairwise elections by the magnitude of the margin of victory. Then, starting with the pairwise election with the largest margin, it “locks in” these results in this order, so that the winner of the current pairwise election must be ranked above the loser in the final ranking—unless this would create a cycle due to previously locked-in results, in which case we move on to the next pairwise election. A similar voting rule was proposed by Schulze [192].

All SCFs mentioned in Section 2.2 (e.g., the top cycle, the uncovered set, and the Banks set) also happen to be Condorcet extensions. This is because the

¹¹Young [230] actually defined his rule using *weak* Condorcet winners (see Exercise 15). Brandt et al. [46] have shown that the hardness result by Rothe et al. [185] carries over to Young’s original definition.

Condorcet criterion can be seen as a very weak variant of expansion consistency: whenever an alternative is chosen in all two-element subsets, then it should also be chosen from the union of all these sets. Many of the proposed Condorcet extensions can be seen as refinements of these SCFs because they always yield elements of, say, the top cycle or the uncovered set. Other prominent Condorcet extensions are Kemeny's rule and Slater's rule (see Section 2.1).

3.1.3 Other Rules

While scoring rules and Condorcet extensions are two important classes of voting rules, many other rules that do not fit in either class have been proposed over the years. Two examples are STV and Bucklin's rule.

STV. We have already mentioned the *STV rule*: it looks for the alternatives that are ranked in first place the least often, removes them from all voters' ballots (so that some of them may now rank a different alternative first), and repeats. The alternatives removed in the last round (which results in no alternatives being left at all) win.

Bucklin's rule. In the (simple version of) *Bucklin's rule*, we first check whether there is any alternative that is ranked first by more than half the voters; if so, this alternative wins. If not, we check whether there are any alternatives that are ranked in either first or second place by more than half the voters; if so, they win. If not, we consider the first three positions, etc. When multiple alternatives cross the $n/2$ threshold simultaneously, it is common to break ties by the margin by which they crossed the threshold.

In order to gain more insight into the huge zoo of voting rules, various axioms that may or may not be satisfied by a voting rule have been put forward. Sometimes a certain set of axioms completely characterizes a single voting rule (such as the SCFs proposed in Section 2.2.2) or an interesting class of voting rules (such as the class of scoring rules in Section 3.1.1). Another stream of research studies the rationalization of voting rules by measuring the *distance* (according to various metrics) of a given preference profile to the nearest preference profile that satisfies certain consensus properties (e.g., being completely unanimous or admitting a Condorcet winner). This approach goes back to Dodgson's voting rule mentioned in Section 3.1.2 and covers many of the rules proposed in this section [161, 99, 100].

3.2 Manipulation

So far, we have assumed that the preferences of all voters are known. In reality, generally the voters need to *report* their preferences. A significant problem is that a voter may be incentivized to report preferences other than her true ones. For example, consider a plurality election between three alternatives, a , b , and c . Consider voter i with preferences $a \succ_i b \succ_i c$. Moreover, suppose that voter i believes that almost nobody else will rank a first, but it will be a close race between b and c . Then, i may be best off casting a vote in which b is ranked first: he has little hope of getting a to win, so he may be better off focusing on ensuring that at least b will win.

One may wonder why manipulation is something to be avoided. First, the possibility of manipulation leads to fairness issues since manipulative skills are usually not spread evenly across the population. Second, energy and resources are wasted on determining how best to manipulate. Third, it makes it difficult to evaluate whether the resulting outcome is in fact one that makes sense with respect to the true preferences (as opposed to the reported ones). As we will see, the question of *how* to manipulate is not only computationally but also conceptually problematic. It raises various fundamental game-theoretic questions and makes it very difficult to make predictions or theoretical statements about election outcomes.¹² There is also a result in the theory of mechanism design known as the *revelation principle* that can be very informally described as saying that anything that can be achieved by a mechanism in which agents play strategically, can also be achieved by a mechanism in which agents are best off telling the truth, underlining again the importance of truthful voting (for more details see the chapter on *Mechanism Design and Auctions* in this volume). Unfortunately, as we shall see, the problem of manipulation cannot be avoided in general as *every* single-valued

¹²One reason for this is that voting games can have many different *equilibria*. For example, in a plurality election, it can be an equilibrium for all voters to vote for either b or c , even though *all* voters rank a first in their true preferences! This is so because if nobody else is expected to vote for a , then it does not make sense to waste one's vote on a . If such an equilibrium seems artificial, imagine a society in which two parties dominate the political scene and put forward candidates b and c , whereas a is a third-party candidate. Of course, there are other equilibria as well, which will in general result in different winners. This makes it difficult to make any predictions about strategic voting. One context in which we can make a sharp game-theoretic prediction of the winner is the one in which the agents vote in sequence, one after the other, observing what the earlier agents have voted (see also Section 4.2). Unfortunately, in this context, paradoxical results can be exhibited where the game-theoretic outcome does not reflect the voters' true preferences well. For more detail, see the work of Desmedt and Elkind [89] and Xia and Conitzer [220].

voting rule for more than two alternatives is susceptible to manipulation.

3.2.1 The Gibbard-Satterthwaite Impossibility

In order to formally capture whether voting rule f can be manipulated by voter i , we initially make the following assumptions. First, since voters just have preferences over single alternatives we assume that f is resolute (i.e., single-valued). And second, we assume that i knows the preferences of all other voters. This latter assumption is not entirely unreasonable in some settings (e.g., decision making in committees) and actually makes all statements about non-manipulability particularly strong because it guarantees non-manipulability even when all preferences are known.

Formally, a resolute voting rule f is *manipulable* by voter i if there exist preference profiles R and R' such that $R_j = R'_j$ for all $j \neq i$ and $f(R') \succ_i f(R)$. (Recall that \succ_i corresponds to the strict part of \succeq_i (not \succeq'_i)). A voting rule is *strategyproof* if it is not manipulable.

Just like an SCF, a voting rule f is *non-dictatorial* if there is no voter i such that for all preference profiles R , $a \in f(R)$ where a is voter i 's most preferred alternative.¹³ Finally, we need a technical condition, even weaker than Pareto-optimality, that ensures that at least three different alternatives can be returned by the voting rule, as follows. A voting rule is *non-imposing* if its image contains all singletons of $\mathcal{F}(U)$, i.e., every single alternative is returned for some preference profile.

Theorem 3 (Gibbard, 1973; Satterthwaite, 1975). *Every non-imposing, strategyproof, resolute voting rule is dictatorial when $|U| \geq 3$.*

Just as for Arrow's Theorem, we will now consider different ways to circumvent this impossibility by calling some of its explicit and implicit assumptions into question.

3.2.2 Restricted Domains of Preferences

One of the implicit assumptions of the Gibbard-Satterthwaite Theorem is that the voting rule needs to be defined for *all* possible preference profiles. An important stream of research has consequently studied which restricted domains of preferences allow for strategyproof voting rules.

¹³For *resolute* voting rules, $a \in f(R)$ obviously implies $\{a\} = f(R)$.

An important observation by Moulin [165] is that in every restricted domain that always admits a Condorcet winner, the SCF that uniquely chooses the Condorcet winner is strategyproof (see Exercise 9). There are many examples of such domains. The best-known among these is the domain of *single-peaked* preferences. Suppose there is a linear ordering $<$ on the alternatives, signifying which alternatives are “smaller” than others. For example, the voters may be voting over what the tax rate should be. In this case, the set of alternatives may be $\{20\%, 30\%, 40\%, 50\%, 60\%\}$, and clearly, $20\% < 30\% < \dots < 60\%$. They may also be voting over possible dates for a deadline, in which case earlier dates could be considered smaller; they may be voting over a location along a road at which to build a building, in which case locations further to the west could be considered “smaller”; or, more abstractly, they could be voting over political candidates, in which case candidates further to the left of the political spectrum could be considered “smaller.”

Imposing an ordering $<$ on the alternatives, in and of itself, of course does not restrict the preferences yet; we must say something about how the preferences *relate* to the order over the alternatives. Preferences are said to be single-peaked with respect to the order $<$ if the following holds: for every voter, as we move away (according to $<$) from the voter’s most-preferred alternative, the alternatives will become less preferred for that voter. Formally, a preference profile R is single-peaked if for every $x, y, z \in U$, it holds that

if $(x < y < z)$ or $(z < y < x)$, then $x \succ_i y$ implies $y \succ_i z$ for every $i \in N$.

When preferences are single-peaked and there is an odd number of voters, there is always a unique Condorcet winner. If we sort voters by $<$ according to their most-preferred alternative (breaking ties arbitrarily), then the $((n+1)/2)$ th voter is called the *median voter*. His top choice is always identical to the Condorcet winner, as was first observed by Black [27]. (The reason that the median voter’s most-preferred alternative c is always the Condorcet winner is simple. Consider any other alternative c' ; without loss of generality, suppose $c' < c$. Then, by single-peakedness, all the voters whose most-preferred alternative is equal to or greater than c will prefer c to c' , and these constitute more than half the voters.) Hence, to determine a Condorcet winner, it suffices to know every voter’s top choice, even though the voters’ preference relations contain more information than just their most-preferred alternatives.

Now, what is the relation of all of this to the Gibbard-Satterthwaite Theorem? The answer is that the median-voter rule, in spite of clearly allowing every alternative the possibility of winning and not being a dictatorial rule, is strategyproof

when we restrict attention to preference profiles that are single-peaked. This follows immediately from the more general result by Moulin [165] that we stated earlier, which says that any Condorcet extension is strategyproof when we only consider profiles with a Condorcet winner. Still, it is instructive to explain the reasons for strategyproofness directly. To do so, consider what a voter who did not get his most preferred alternative elected could do. Suppose the winner a is “smaller” than his own top choice b . If he manipulates and instead of truthfully declaring b as his top choice decides to report a “smaller” alternative b' , then either the winner will not change or the winner will become even “smaller” and thus even less attractive. On the other hand, if he reports a “larger” alternative b'' instead, then he will not affect the median (and thus the winner) at all. Hence, any form of manipulation will either damage his interests or have no effect at all. The exact same argument would continue to hold even if instead of choosing the median, or 50th-percentile, voter, we chose the (say) 60th-percentile voter, even though this clearly would not necessarily choose the Condorcet winner. The argument is also easily modified to prove the stronger property of *group-strategyproofness* (where a group of agents can join forces in attempting to manipulate the outcome).

Singled-peakedness has also been studied from a computational point of view. It is very easy to check whether a preference profile is single-peaked according to a specific given ordering $<$. However, it is less obvious whether it can be checked efficiently whether a preference profile is single-peaked according to *some* ordering $<$. Building on previous work by Bartholdi, III and Trick [18], Escoffier et al. [106] proposed a linear-time algorithm for this problem. In other work, Conitzer [63] and Farfel and Conitzer [116] investigated how to elicit the voters’ preferences by asking as few queries as possible when preferences are known to be single-peaked (with the latter paper focusing on settings where agents have most-preferred *ranges* of alternatives). The computational hardness of manipulation (which will be introduced in the next section) for other voting rules than median voting has also been examined in the context of single-peaked preferences [213, 111, 46].

Another important domain of restricted preferences is that of *value-restricted* preferences which also guarantees the existence of a Condorcet winner and subsumes many other domains such as that of single-peaked preferences [197, 201].

3.2.3 Computational Hardness of Manipulation

The positive results for restricted preferences discussed above are encouraging for settings where we can expect these restrictions to hold. Unfortunately, in many

settings we would *not* expect them to hold. For example, while placing political candidates on a left-to-right spectrum may give us some insight into what the voters' preferences are likely to be, we would still expect many of their preferences to be not exactly single-peaked: a voter may rank a candidate higher because the candidate is especially charismatic, or perhaps voters are somewhat more sophisticated and they really consider two spectra, a social one and an economic one. This is perhaps the main downside of the approach of restricting the voters' preferences: we generally have no control over whether the preferences actually satisfy the restriction, and if they do not, then there is little that we can do.

We now discuss another approach to circumventing impossibility results such as the Gibbard-Satterthwaite Theorem. Here, the idea is that the mere theoretical *possibility* of manipulation need not be a problem if in practice, opportunities for manipulation are computationally too hard to *find*. So, how hard is it to find effective manipulations? For this, we first need to clearly define the manipulation problem as a computational problem. The best-known variant is the following, which takes the perspective of a single voter who (somehow) already knows all the other votes, and wishes to determine whether he can make a particular alternative the winner.

Definition 4. *In the manipulation problem for a given resolute voting rule, we are given a set of alternatives, a set of (unweighted) votes, and a preferred alternative p . We are asked whether there exists a single vote that can be added so that p wins.*¹⁴

One may object to various aspects of this definition. First of all, one may argue that what the manipulator seeks to do is not to make a given alternative the winner, but rather to get an alternative elected that is as high in his true ranking as possible. This does not pose a problem: if the manipulator can solve the above problem, he can simply check, for every alternative, whether he can make that alternative win, and subsequently pick the best of those that can win. (Conversely, to get an alternative elected that is as high in his true ranking as possible, he needs to check first of all whether he can make his most-preferred alternative win, which comes down to the above problem.) Another objection is that the manipulator generally does not know the votes of all the other voters. This is a reasonable objection, though it should be noted that as long as it is *possible* that the manipulator knows

¹⁴Often, the problem is defined for irresolute voting rules; in this case, the question is either whether p can be made one of the winners, or whether p can be made the unique winner. These questions can be interpreted to correspond to the cases where ties are broken in favor of p , and where they are broken against p , respectively.

the votes of all the other voters, the above problem remains a special case (and thus, any (say) NP-hardness results obtained for the above definition still apply.

Inspired by early work by Bartholdi, III et al. [19], recent research in computer science has investigated how to use computational hardness—primarily NP-hardness—as a barrier against manipulation [see, e.g., 68, 74, 98, 129, 110]. Finding a beneficial manipulation is known to be NP-hard for several rules, including second-order Copeland [19], STV [17], ranked pairs [224], and Nanson and Baldwin’s rules [166]. Many variants of the manipulation problem have also been considered. In the *coalitional* manipulation problem, the manipulators can cast multiple votes in their joint effort to make p win. Because the single-manipulator case is a special case of this problem where the coalition happens to have size 1, this problem is NP-hard for all the rules mentioned above. However, other rules are also NP-hard to manipulate in this sense, including Copeland [109, 115], maximin [224], and Borda [25, 85]. Finally, in the *weighted* version of this problem, weights are associated with the voters, including the manipulators (a vote of weight k counts as k unweighted votes). Here, many rules become NP-hard to manipulate even when the number of alternatives is fixed to a small constant [74, 129, 166].

In the *destructive* version of the problem, the goal is not to make a given alternative a win, but rather to make a given alternative a *not* win [74]. For contrast, the regular version is called the *constructive* version. If the constructive version is easy, then so is the destructive version, because to solve the destructive version it suffices to solve the constructive version for every alternative other than a ; but in some cases, the destructive version is easy while the constructive version is not.

Computational hardness has also been considered as a way of avoiding other undesirable behavior. This includes *control* problems, where the chair of the election has (partial) control over some aspects of the election (such as which alternatives are in the running or which voters get to participate) and tries to use this to get a particular alternative to win [16, 132, 179, 75, 113]. Another example is the *bribery* problem, where some interested party attempts to bribe the voters to bring about a particular outcome [108, 112, 113].

One downside of using NP-hardness to prevent undesirable behavior—whether it be manipulation, control, or bribery, but let us focus on manipulation—is that it is a *worst-case* measure of hardness. This means that if the manipulation problem is NP-hard, it is unlikely that there is an efficient algorithm that solves *all* instances of the manipulation problem. However, there may still be an efficient algorithm that solves *many* of these instances fast. If so, then computational hardness provides only partial protection to manipulation, at best. It would be much

more desirable to show that manipulation is *usually* hard. Recent results have cast doubt on whether this is possible at all. For instance, it was shown that, when preferences are single-peaked many of the manipulation problems that are known to be NP-hard for general preferences, become efficiently solvable [111, 46]. In other work on certain distributions of unrestricted preferences, both theoretical and experimental results indicate that manipulation is often computationally easy [e.g., 71, 176, 177, 218, 217, 214]. Extending a previous result by Friedgut et al. [119], Isaksson et al. [139] have recently shown that efficiently manipulable instances are ubiquitous under fairly general conditions.

Theorem 4 (Isaksson et al., 2010). *Let f be a neutral resolute voting rule and assume that preferences are uniformly distributed. The probability that a random preference profile can be manipulated by a random voter by submitting random preferences is at most polynomially small in $|U|$ and n .*

As a consequence, for efficiently computable, neutral, and resolute voting rules, a manipulable preference profile with a corresponding manipulation can easily be found by repeated random sampling. The current state of affairs on using computational hardness to prevent manipulation is surveyed by Faliszewski and Procaccia [107].

3.2.4 Probabilistic Voting Rules

Perhaps the only weakness of the Gibbard-Satterthwaite Theorem is that it is restricted to *resolute* voting rules [see, e.g., 206]. As we have seen in Section 3, resoluteness is at variance with elementary fairness conditions such as anonymity and neutrality. The most natural way to break ties yielded by an irresolute voting rule that comes to mind is to pick a single winner at random according to some probability distribution. In order to formalize this, Gibbard [125] proposed an extension of voting rules called *social decision schemes (SDSs)*, which map preference profiles to probability distributions (so-called lotteries) over alternatives.

Of course, the introduction of lotteries raises the question of how voters compare lotteries with each other. The standard approach chosen by Gibbard [125] and subsequent papers [e.g., 126, 11] is to use an expected utility model. In this context, an SDS is strategyproof if, for any utility function that the voter may have over the alternatives, the voter is best off reporting the ordering of the alternatives that corresponds to his true utility function.

Standard examples of non-manipulable SDSs are *random dictator* rules, in which the most preferred alternative of a randomly selected voter is chosen ac-

cording to a probability distribution that does *not* depend on the voters' preferences. While these rules are clearly fairer than rules with a fixed dictator, they are still not entirely desirable. Unfortunately, when requiring *non-imposition*, i.e., every alternative may be chosen with probability 1 under *some* circumstances (e.g., when all voters unanimously agree), random dictatorships are the only non-manipulable SDSs.

Theorem 5 (Gibbard, 1977; Hylland, 1980). *Every non-imposing, non-manipulable SDS is a random dictatorship when $|U| \geq 3$.*

While this might appear like the natural equivalent of the Gibbard-Satterthwaite Theorem, it may be argued that non-imposition is rather strong in this context. Gibbard [125] provides a different characterization that uses so-called *duple rules*, in which the outcome is always restricted to two randomly chosen alternatives (e.g., by applying the majority rule to a random pair of alternatives), which no longer seems so unreasonable. Following along these lines, Barberà [12] and Procaccia [175] provide further examples and characterizations. However, all of these SDSs require an extreme degree of randomization.¹⁵

3.2.5 Irresolute Voting Rules

The definition of manipulability for SDSs rests on strong assumptions with respect to the voters' preferences. In contrast to the traditional setup in social choice theory, which typically only involves ordinal preferences, this model relies on the axioms of von Neumann and Morgenstern in order to compare lotteries over alternatives. The gap between the Gibbard-Satterthwaite Theorem for resolute voting rules and Gibbard's theorem for social decision schemes has been filled by a number of impossibility results for *irresolute* voting rules with varying underlying notions of how to compare sets of alternatives with each other [see, e.g., 206, 13].

How preferences over sets of alternatives relate to or depend on preferences over individual alternatives is a fundamental issue that goes back to the foundations of decision making. There is no single correct answer to this question. Much depends on the particular setting considered, the nature of the alternatives, and what we can assume about the personal inclinations of the agent entertaining the preferences. In the context of social choice the alternatives are usually interpreted as mutually exclusive candidates for a unique final choice. For instance, assume a

¹⁵An important extension of this model studies SDSs in which the von Neumann-Morgenstern utility functions of the voters rather than their preference relations are aggregated [see, e.g., 137, 14, 96].

voter prefers a to b , b to c , and—by transitivity— a to c . What can we reasonably deduce from this about his preferences over the subsets of $\{a, b, c\}$? It stands to reason to assume that he would strictly prefer $\{a\}$ to $\{b\}$, and $\{b\}$ to $\{c\}$. If a single alternative is eventually chosen from each choice set, it is safe to assume that he also prefers $\{a\}$ to $\{b, c\}$, but whether he prefers $\{a, b\}$ to $\{a, b, c\}$ already depends on (his knowledge about) the final decision process. In the case of a lottery over all pre-selected alternatives according to a known *a priori* probability distribution with full support, he would prefer $\{a, b\}$ to $\{a, b, c\}$.¹⁶ This assumption is, however, not sufficient to separate $\{a, b\}$ and $\{a, c\}$. Based on a sure-thing principle which prescribes that alternatives present in both choice sets can be ignored, it would be natural to prefer the former to the latter. Finally, whether the voter prefers $\{a, c\}$ to $\{b\}$ depends on his attitude towards risk: he might be an *optimist* and hope for his most-preferred alternative, or a *pessimist* and fear that his least-preferred alternative will be chosen. One of the most influential negative results for irresolute rules is the *Duggan-Schwartz impossibility* [92].

Theorem 6 (Duggan and Schwartz, 2000). *Every non-imposing, non-dictatorial voting rule can be manipulated by an optimist or pessimist when $|U| \geq 3$.*

However, for weaker (incomplete) preference relations over sets more positive results can be obtained [e.g., 39, 41]. Brandt [39], for instance, has shown that the minimal covering set and the bipartisan set (mentioned in Section 2.2.2) are non-manipulable when one set of alternatives is preferred to another if and only if everything in the former is preferred to everything in the latter.

3.3 Possible and Necessary Winners

Two commonly studied computational problems in voting are the *possible winner problem* and the *necessary winner problem* [142]. The input to these problems is a partially specified profile of votes and a distinguished alternative c ; we are asked whether there exists some completion of the profile that results in c winning (possible winner) or whether c will in fact win no matter how the profile is completed (necessary winner).¹⁷

There are several important motivations for studying these problems. One derives from the *preference elicitation problem*, where we repeatedly query voters

¹⁶The posterior distribution is obtained by conditioning on the selected subset. This rules out inconsistent lotteries like always picking b from $\{a, b\}$ and a from $\{a, b, c\}$.

¹⁷Other work has considered this problem in the setting where instead of uncertainty about the profile, there is uncertainty about the voting rule [150].

for parts of their preferences until we know enough to determine the winner. The necessary winner problem is of interest here, because at an intermediate stage in the elicitation process, we will know the profile partially and may wish to know whether we can safely terminate and declare c the winner. (The computational problem of determining whether elicitation is done was explicitly studied by Conitzer and Sandholm [67].) The possible and necessary winner problems also have as a special case the problems faced by a set of manipulators who know the subprofile of the other voters (when they wish to make a given alternative win or prevent a given alternative from winning, respectively). This latter observation allows us to easily transfer hardness results for the manipulation problem to the possible and necessary winner problems. In general, however, the possible and necessary winner problems can be even harder than the corresponding manipulation problems, because the possible and necessary winner problems generally also allow individual votes to be only partially specified (which makes sense under the elicitation interpretation, because we may so far have asked voters to compare only certain pairs of alternatives, and not others).¹⁸

A natural way for expressing partial knowledge about the voters' preferences is to have a *partial order* over the alternatives associated with every voter. The idea is that we know that the voters preferences must be some linear order that extends that partial order. The computational complexity of this problem for various voting rules has been determined by Xia and Conitzer [221]. The possible winner problem is NP-complete for rules including STV, scoring rules including Borda and k -approval,¹⁹ Copeland, maximin, Bucklin, and ranked pairs. The necessary winner problem is coNP-complete for all these except scoring rules, maximin, and Bucklin, for which it can be solved in polynomial time. For plurality and anti-plurality, both problems can be solved in polynomial time.

4 Combinatorial Domains

So far we have presented the classical mathematical framework for studying different variants of the problem of social choice and we have seen examples of questions regarding the computational properties of this framework. Next, we will

¹⁸Recent work has also studied a version of the manipulation problem where the profile of nonmanipulator votes is only partially known to the manipulator [79], which is another problem that is closely related to the possible/necessary winner problem.

¹⁹The complexity of the possible winner problem for scoring rules has been completely characterized by Betzler and Dorn [22] and Baumeister and Rothe [21].

consider a social choice problem where computational considerations already play a central role at the level of *defining* the formal framework to study this problem. The problem in question is the problem of *social choice in combinatorial domains*. To simplify matters, we will focus specifically on *voting* in combinatorial domains.

Let us begin with an example. Suppose three agents need to agree on a menu for dinner. The options for the starter are *salad* and *oyster*; the options for the main course are *trout* and *veal*; and the options for the wine are *red* and *white*. The favorite menus of our three agents are as follows.

Agent 1: *salad-trout-white*

Agent 2: *salad-veal-red*

Agent 3: *oyster-veal-white*

Agent 1 likes trout and naturally wants to combine this with a white wine; agent 2 likes veal (which may be paired with either red or white wine) and has a preference for red wine; and agent 3 likes oyster and veal, which calls for a white wine. Now, what menu should our agents choose as a group, and how should they make that choice? Maybe the most natural approach is to use the *plurality rule* on each of the three issues: there is a majority for salad, there is a majority for veal, and there is a majority for white wine. That is, the group menu will be *salad-veal-white*. But this very conceivably could be one of the worst possible choices for our agents: like agent 2, they may very well all prefer to have a red wine with salad and veal.

What went wrong here? The problem is that the preferences of the agents over the choices made for each of the three issues are not *independent*. For instance, our little story suggested that for all of them their preferred choice of wine depends on what starter and main course they will actually get served. But voting issue-by-issue completely ignores this dependency, and so we should not be too surprised if we get a paradoxical outcome.

Note also that the next most obvious approach, which would be to directly vote on full menus does not work very well either. If we ask each agent only for his most preferred menu (as we have done above), we will typically get three different answers, and the best we can do is to randomly select one of the three. We could refine this approach further, and ask, say, for their five most preferred menus and apply, say, the Borda rule. This might lead to an acceptable solution in our little example, but imagine we are dealing with a choice problem with 10 binary issues and thus $2^{10} = 1024$ alternatives: the most preferred alternatives of our three agents might very well be entirely disjoint again.

A full description of our example should actually list the full preferences of each of our three agents over the *combinatorial domain* $D = \{salad, oyster\} \times \{trout, veal\} \times \{red, white\}$, i.e., over a set of eight alternatives. Note that the number of alternatives is *exponential* in the number of issues. But this means that even for examples with a slightly larger number of issues it can quickly become practically infeasible for the agents to rank all the alternatives and communicate this ranking. That is, there is a fundamental computational challenge hidden at the very heart of voting in combinatorial domains: even a small problem description immediately gives rise to a very large choice problem.

In our little example there actually is a good solution: For all three agents, their preferences regarding the wine depend on the choices made for the starter and the main course, while their preferences for those two issues do not depend on anything else (we have not actually described our example in enough detail before to be sure about the latter fact, but let us now assume that this is indeed the case). We can use these dependencies to determine a good order in which to vote on each of the three issues *in sequence*. As long as we vote on the wine at the very end, there will not be any paradoxical outcome (nor will there be any computational difficulty).²⁰

So, if we first use the plurality rule to choose a starter and a main course, our agents are likely to choose the salad and the veal. If we then fix these choices and ask the agents to vote on the wine, they will select the red wine, yielding an outcome (*salad-veal-red*) that is ideal for agent 2 and not unreasonable for the other two.

The kind of paradox we have seen has long been observed and studied in political science, typically under the name of “multiple-election paradoxes” [35]. As a problem that is inherently computational in nature it was first formulated by Lang [148].

As the representation of an agent’s preferences plays a central role in social choice in combinatorial domains, we will first review the most important knowledge representation languages that have been used in the literature to this end. We will then focus on two types of promising approaches: sequential voting and voting by means of compactly represented preferences.

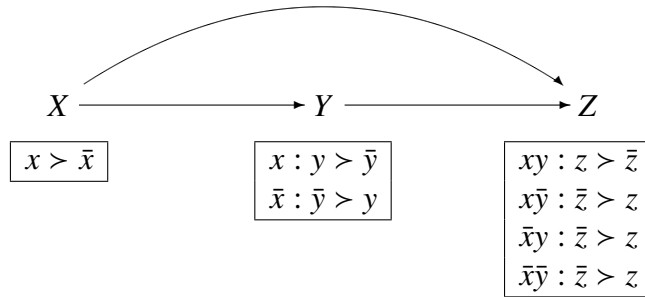
²⁰This is assuming that agents do not vote strategically; we will discuss this point more at the end of Section 4.2.

4.1 Preference Representation

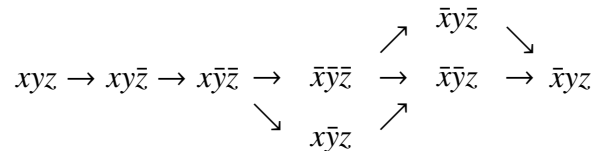
Suppose we want to model an agent’s preferences over a combinatorial domain defined by ℓ binary variables, i.e., over a domain with 2^ℓ alternatives. There are $(2^\ell)!$ different linear orders that might represent the agent’s preferences, and there are even more possibilities if we want to also consider weak or partial orders. Encoding such a linear order thus requires at least $\log(2^\ell!)$, i.e., $O(\ell \cdot 2^\ell)$, bits. If we use an *explicit representation* that specifies for each pair of alternatives whether or not our agent prefers the first of them over the second, we even need $O(2^\ell \cdot 2^\ell)$ bits (one per pair). This will generally not be feasible in practice. Instead, we require a *compact* preference representation language that will allow us to model those preference structures that we can expect to occur in a given application scenario using short expressions in that language. When choosing such a language, we should consider its *expressive power* (which preference structures can it express?), its relative *succinctness* (can it do so using significantly less space than a given rival language?), its *complexity* (how hard is it to reason about preferences expressed in the language?), its *elicitation-friendliness* (does it support efficient elicitation of preferences from the agents?), and its *cognitive adequacy* (is it a “natural” form of describing preferences?) [54].

The most widely used language for compact preference representation used in computational social choice are *conditional preference networks*, or CP-nets [31].

The basic idea is to induce a preference order from statements of the form “if condition C holds, then—everything else being equal—I prefer variable X to take value x rather than value \bar{x} ”. A CP-net is based on a directed graph on the set of variables defining the combinatorial domain in question. Every vertex in the graph is annotated with a table that specifies, for each possible instantiation of the variables corresponding to the parents of that vertex, a preference order over the possible values of the variable corresponding to that vertex. Let us consider an example. Suppose our domain is defined by means of three variables: X (with possible values x and \bar{x}), Y (with possible values y and \bar{y}), and Z (with possible values z and \bar{z}). A CP-net for this domain might look like this:



A CP-net induces a partial order: if two alternatives differ only in the instantiation of a single variable, then we can look up the corresponding entry in the table for that variable to find how the two alternatives should be ranked. The full partial order is the transitive closure of the relations we obtain by interpreting the individual preference statements in this manner. For instance, given the CP-net above, we prefer $xy\bar{z}$ to $x\bar{y}\bar{z}$, because the two differ only in their assignment to Y , and the first statement in the table for variable Y says that when $X = x$, then we should prefer $Y = y$ over $Y = \bar{y}$, everything else being equal (i.e., $Z = \bar{z}$ in both cases). The full preference relation induced by the CP-net above is the following partial order (where an arrow represents $>$ and the rankings obtained by transitivity are not shown explicitly):



Note that, for instance, $\bar{x}\bar{y}\bar{z}$ and $x\bar{y}\bar{z}$ are *incomparable*: the CP-net does not specify which of the two the agent prefers.

Another important family of languages for preference representation is that of *prioritized goals* [148, 81]. Prioritized goals are applicable when each of the variables defining the combinatorial domain has exactly two possible values (e.g., *true* and *false*, or 1 and 0). The basic idea is to describe the goals of the agent whose preferences we are modeling as formulas of propositional logic. For example, the formula $X \vee Y$ expresses the goal of having at least one of the variables X and Y take the value *true*, while $X \rightarrow \neg(Y \wedge Z)$ says that whenever X is *true*, then it should not be the case that both Y and Z are *true* as well. Usually not all goals will be satisfiable. An agent can indicate the importance of each of his goals by labeling it with a number, its priority level (suppose a higher number indicates

higher importance). Different interpretations of this kind of language are possible. One choice is the *lexicographic* interpretation, under which we prefer alternative a to alternative b if there exists a k such that for every priority level above k both a and b satisfy the same number of goals, while a satisfies more goals of priority level k than b does. For example, if an agent has the goals X and $\neg Y$ and the former has higher priority than the latter, then this induces the preference order $x\bar{y} > xy > \bar{x}\bar{y} > \bar{x}y$.

Both CP-nets and prioritized goals define preferences that are ordinal: either linear or weak orders, as commonly used in classical social choice theory, or partial orders and preorders, which can be particularly appropriate for applications in multiagent systems, where we may want to model explicitly the fact that an agent has bounded rationality and is lacking either the computational resources or the necessary information to completely rank all possible pairs of alternatives. In Section 5.1, in the context of discussing fair division problems, we will see further examples of preference representation languages, namely languages for modeling cardinal preferences (i.e., valuation functions).

4.2 Sequential Voting

In the example above, we already mentioned the idea of voting over the issues one at a time, in sequence. This is a very natural idea and requires relatively little communication. A downside of sequential voting is that an agent's preferences for the current issue may depend on issues on which we have not yet decided. (Above, avoiding such a situation was our motivation for deciding on the wine last.) CP-nets allow us to formalize this idea. We say that a CP-net is *legal* for an order over the issues if its graph does not have any edges pointing from issues that are later in the order to ones that are earlier (which immediately implies that the graph is acyclic). As a result, if all the agents' CP-nets are legal for the order in which we vote over issues, then each agent's CP-net will always unambiguously specify the agent's preferences over the current issue, because we will have already decided on the values of the issues on which these preferences depend. This also means that the agents do not actually need to vote on each issue separately; they can just submit their CP-nets, and then leave. These ideas and the properties of sequential voting are discussed in detail by Lang and Xia [149].

It is of course still possible to force agents to use sequential voting even if their preferences for earlier issues do depend on later issues, but in this case it is no longer clear how they should vote. One possibility is to assume that agents vote strategically, thinking ahead towards what is likely to happen regarding later

issues (which may also depend on how they vote on the current issue). It should be noted that this will, in general, change how the agents vote even when their preferences for earlier issues do not depend on later issues. Unfortunately, as we have discussed earlier, even when there is only a single issue, strategic voting is quite complicated when that issue can take three or more possible values—for example, the corresponding game has multiple equilibria. On the other hand, if we assume that each issue can take only two possible values and that the agents’ true preferences are common knowledge, then it is clear how to vote strategically. This is because in the last round (when voting over the last issue) the agents are effectively voting over the two remaining alternatives, so each agent is best off voting for his preferred one; based on this, in the second-to-last round, the agents can predict which alternative will end up chosen in the last round as a function of which value the current (second-to-last) issue ends up taking, so effectively the agents are deciding between the corresponding two alternatives; and so on. Specifically, this means that under these circumstances strategic sequential voting is bound to result in the election of the Condorcet winner, whenever it exists [145].

On the other hand, Xia et al. [226] show that, unfortunately, for some profiles without a Condorcet winner, strategic sequential voting results in very poor outcomes; in fact, this happens even when the agents’ preferences for earlier issues never depend on the agents’ preferences for later issues, because they will not necessarily vote truthfully. (Incidentally, the strategic sequential voting process is a special case of *multistage sophisticated voting* [159, 162, 128].) Xia et al. [226] also show that the outcome can be very sensitive to the order in which the issues are voted on, potentially giving the agenda setter a significant amount (or even complete) control over the outcome. The complexity of this control problem has been studied in a nonstrategic context [75].

4.3 Voting with Compactly Represented Preferences

Considerations of strategic voting aside, sequential voting is an appealing procedure when each agent’s preferences are represented by a CP-net that is legal with respect to the same ordering. But what if this is not the case? For one, the agents may require different orders. For example, consider one agent who prefers veal to trout regardless of which wine is chosen, but whose preferred wine depends on which meal is served; and another agent who prefers red to white wine regardless of which meal is served, but whose preferred meal depends on which wine is served. For the former agent, it would be ideal to vote on the meal first, but for the latter, it would be ideal to vote on the wine first. Clearly,

there is no way to order the issues that will make both agents comfortable voting on the first issue without knowing what value the second issue will take. Moreover, it is not necessarily the interaction between multiple agents' preferences that causes trouble: it is even possible for a single agent's preferences to conflict with the idea of sequential voting. For example, consider an agent who mostly cares that the wine fits the meal, and ranks the different meal combinations $(trout, white) > (veal, red) > (veal, white) > (trout, red)$. For this agent, no ordering of the issues is satisfactory, because his preference for each issue depends on the other issue—his CP-net is cyclic.

How can we address such preferences, without falling back on making the agents rank all the exponentially many alternatives, but rather by making use of a compact preference representation language, such as CP-nets? This problem has been introduced by Lang [148] under the name of “combinatorial voting”, i.e., voting by means of ballots that are compactly represented preference structures. Unfortunately, the computational complexity of this approach is often prohibitively high. For example, Lang [148] shows that computing the election winners when each voter specifies his preferences using the language of prioritized goals and (a suitable generalized form of) the plurality rule is used is coNP-hard, even when each voter only states a single goal. Similarly, deciding whether there exists a Condorcet winner is coNP-hard under the same conditions. For languages that can express richer preference structures, the complexity of winner determination will typically be beyond NP.

One useful property of preferences represented by a CP-net is that, if we hold the values of all but one issue fixed, then the CP-net specifies the agent's preferences over that remaining issue. While it is not computationally efficient to do so, conceptually, we can consider, for every issue and for every possible setting of the other issues, all agents' preferences. We can then choose winners based on these “local elections” [223, 153, 78]. For example, we can look for an alternative that defeats all of its neighboring alternatives (that is, the alternatives that differ on only one issue from it) in pairwise elections. Of course, there may be more than one such alternative, or none. The maximum likelihood approach mentioned earlier in this chapter has also been pursued in this context [225].

Developing practical algorithms for voting in combinatorial domains is one of the most pressing issues on the research agenda for computational social choice in the near future.

5 Fair Division

So far we have discussed social choice in its most general and abstract form, as the problem of choosing one or several “alternatives”, or as the problem of ranking them. An alternative might be a candidate to be elected to political office or it might be a joint plan to be executed by a group of software agents. In principle, it might also be an allocation of resources to a group of agents. In this section, we specifically focus on this latter problem of *multiagent resource allocation*. To underline our emphasis on fairness considerations we shall favor the term *fair division*. In the economics literature, the broad research area concerned with determining a fair and economically efficient allocation of resources in society is known as *welfare economics*. We will introduce some of the fundamental concepts from this literature, discuss them from an algorithmic point of view, and review their relevance to multiagent systems.

Fair division differs from the other types of social choice problems discussed in this chapter in at least two important respects:

1. Fair division problems come with a rich *internal structure*: alternatives are allocations of goods to agents and an agent’s preferences are usually assumed to only depend on their own bundle.
2. In the context of fair division problems, preferences are usually assumed to be *valuation functions*, mapping allocations to numerical values, rather than binary relations for ranking allocations.

Below (in Section 5.1) we will therefore begin by reviewing preference representation languages for compactly modeling such valuation functions.

First, however, we need to fix the type of goods to be allocated. The main line of differentiation is between *divisible* and *indivisible* goods. A classical example of the former scenario is the fair division of a cake. While there have been a number of contributions to the cake-cutting literature in theoretical computer science and more recently also in artificial intelligence, to date, most work in multiagent systems has concentrated on indivisible goods. We shall therefore only give one example here, which is illustrative of the simple and elegant solutions that have been obtained in the field of cake-cutting. Consider the *moving-knife procedure* due to Dubins and Spanier [91]:

A referee moves a knife across the cake, from left to right. Whenever an agent shouts “stop”, he receives the piece to the left of the knife and leaves.

Under standard assumptions on the agents’ valuation functions (namely, *continuity* and *additivity*), this procedure is *proportional*: it guarantees each agent at least $\frac{1}{n}$ of the full value of the cake, according to their own valuation, whatever the other agents may do (where n is the number of agents). To see this, observe that you are free to shout “stop” when the knife reaches a point where the piece to be cut would be exactly $\frac{1}{n}$ according to your valuation; and if another agent shouts “stop” earlier, then that only means that he will leave with a piece that you consider to be of less value than $\frac{1}{n}$. The books by Brams and Taylor [37] and by Robertson and Webb [182] both provide excellent expositions of the cake-cutting problem.

For the remainder of this section we will focus on the problem of fairly allocating a set of indivisible goods to a group of agents. After introducing several languages for representing preferences in this context, we define the most important criteria for measuring fairness and economic efficiency, and we review examples of work in computational social choice concerning the complexity of computing a fair allocation and designing protocols that can guarantee convergence to a fair allocation. For a more extensive review of the variety of multiagent resource allocation problems and computational aspects of fairness than is possible here we refer to the survey article by Chevaleyre et al. [54].

5.1 Preference Representation

Let G be a finite set of indivisible *goods*, with $\ell = |G|$. Each agent may receive any subset of G . The preferences of agent $i \in N$ are given by means of a *valuation function* $v_i : 2^G \rightarrow \mathbb{R}$, mapping every bundle he might receive to the value he assigns to it. Valuation functions are often assumed to be *monotonic*, i.e., for any two sets of goods S and S' , it will be the case that $v_i(S) \leq v_i(S')$ whenever $S \subseteq S'$. This assumption is also known as *free disposal*. For many applications it makes sense to assume that valuation functions are monotonic, while for others we also need to be able to model undesirable goods.

An *explicit representation* of a valuation function v_i will often not be feasible in practice, as it requires us to specify a list of 2^ℓ numbers. However, if valuations are “well-behaved” in the sense of exhibiting some structural regularities, then a compact representation using a suitable preference modeling language will often be possible.

A powerful family of languages, closely related to the prioritized goal languages discussed in Section 4.1, is based on *weighted goals*. This language originates in the work on penalty logic of Pinkas [172] and variants of it have been used in many areas of artificial intelligence and elsewhere. Its relevance for pref-

erence representation in the context of social choice has first been recognized by Lafage and Lang [146]. The basic idea is again to have an agent express his goals in terms of formulas of propositional logic and to assign numbers, or weights, to these goals to indicate their importance. We can then aggregate the weights of the goals satisfied by a given alternative to compute the value of that alternative. The most widely used form of aggregation is to take the *sum* of the weights of the satisfied goals. For example, suppose G is a set of three goods, associated with the propositional variables p , q and r . An agent providing the weighted goals $(p \vee q, 5)$ and $(q \wedge r, 3)$ expresses the following valuation function:

$$\begin{array}{rcl}
 v(\emptyset) & = & 0 \\
 v(\{p\}) & = & 5 \\
 v(\{q\}) & = & 5 \\
 v(\{r\}) & = & 0 \\
 v(\{p, q\}) & = & 5 \\
 v(\{p, r\}) & = & 5 \\
 v(\{q, r\}) & = & 8 \\
 v(\{p, q, r\}) & = & 8
 \end{array}$$

That is, obtaining one of p and q (or both) has value 5 for him, and in case he obtains the latter, obtaining r on top of it results in an additional value of 3.

By putting restrictions on the kinds of formulas we want to admit to describe goals, we can define different languages. For instance, we may only permit conjunctions of atomic formulas, or we may only allow formulas of length at most 3, and so forth. Different such languages have different properties, in view of their *expressive power*, in terms of their *succinctness* for certain classes of valuations functions, and regarding the *computational complexity* of basic reasoning tasks, such as computing the most preferred bundle for an agent with a given set of weighted goals. For full definitions and a detailed analysis of these properties, we refer to the work of Uckelman et al. [211].

Weighted goal languages are closely related to other languages to be found in the literature. For instance, *k-additive functions*, studied in measure theory [127], correspond to the weighted goal language we obtain when the only admissible logical connective is conjunction and when each formula may involve at most k propositional variables [211]. In cooperative game theory, *marginal contribution nets* [138], a language for modeling coalitional games, correspond to the language of conjunctions of literals of arbitrary length.²¹ Weighted goal languages have also been studied for other forms of aggregation than summing up the weights of the satisfied goals [146, 210].

²¹In some expositions of marginal contributions nets the restriction to conjunctions of literals is not imposed, in which case we obtain the general language of weighted goals (see, e.g., the chapter on *Computational Coalition Formation* in this volume).

Preference representation languages also play an important role in the literature on *combinatorial auctions* [82], where they are known as *bidding languages*. In an auction, each bidder has to describe his valuation of the goods on sale to the auctioneer, i.e., a bid amounts to the specification of a valuation function (whether or not the bidder does so *truthfully* is irrelevant for the representation problem at hand). The idea of using weighted goals has been used also in this domain [30]. The most widely used basic bidding languages, however, belong to the *OR/XOR family*. An *atomic bid* is a bundle of goods together with a price, e.g., $(\{p, q\}, 7)$. Each bidder can provide any number of atomic bids. Under the *OR-language*, the value of a bundle for the bidder is the maximum price that we can obtain by assigning each item in the bundle to (at most) one atomic bid and summing up the prices for those atomic bids that are covered completely by this assignment. For example, given the bid $(\{p, q\}, 7)$ OR $(\{p, r\}, 5)$ OR $(\{r\}, 3)$, the value of the bundle $\{p, q, r\}$ is $7 + 3 = 10$. Under the *XOR-language* the value of a bundle is the price of the most valued atomic bid it can cover. That is, the XOR-language is like the explicit representation mentioned earlier (together with an implicit monotonicity assumption). Combinations of OR and XOR have also been studied. Full definitions and results regarding the expressive power and succinctness of different languages are available in a review article by Nisan [169].

Yet another option is to think of a valuation function as a *program* that takes bundles as inputs and returns values as output. In the context of fair division, this idea has been explored in the work of Dunne et al. [94].

While most work in fair division assumes that preferences are given in the form of valuation functions the problem of fairly dividing goods over which agents have ordinal preferences is also interesting. CP-nets, the most important language for research on voting in combinatorial domains is only of very limited interest here, because CP-nets cannot express most *monotonic* preferences. A possible alternative are so-called *conditional importance networks*, or CI-nets [34].

5.2 Fairness and Efficiency

What makes a *fair* allocation of resources? More generally, what makes a *good* allocation? Next we shall review several proposals for measuring the quality of an allocation. The first set of proposals is based on the idea of a *collective utility function*. Any given allocation yields some utility $u_i \in \mathbb{R}$ for agent i . This utility will usually be the result of applying agent i 's valuation function to the bundle he receives under the allocation in question. Now we can associate an allocation with a *utility vector* $(u_1, \dots, u_n) \in \mathbb{R}^n$.

Definition 5. A collective utility function (CUF) is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

That is, a CUF returns a single collective utility value for any given utility vector (which in turn we can think of as being generated by an allocation). This collective utility is also referred to as the *social welfare* of the corresponding allocation. The following are the most important CUFs studied in the literature:

- Under the *utilitarian* CUF, $f_u(u_1, \dots, u_n) := \sum_{i \in N} u_i$, i.e., the social welfare of an allocation is the sum of the utilities of the individual agents. This is a natural way of measuring the quality of an allocation: the higher the *average* utility enjoyed by an agent, the higher the social welfare. On the other hand, this CUF hardly qualifies as *fair*: an extra unit of utility awarded to the agent currently best off cannot be distinguished from an extra unit of utility awarded to the agent currently worst off. Note that authors simply writing about “social welfare” are usually talking about utilitarian social welfare.
- Under the *egalitarian* CUF, $f_e(u_1, \dots, u_n) := \min\{u_i \mid i \in N\}$, i.e., the social welfare of an allocation is taken to be the utility of the agent worst off under that allocation. This CUF clearly does focus on fairness, but it is less attractive in view of economic efficiency considerations. In the special case where we are only interested in allocations where each agent receives (at most) one item, the problem of maximizing egalitarian social welfare is also known as the *Santa Claus Problem* [9].
- A possible compromise is the *Nash CUF*, under which $f_n(u_1, \dots, u_n) := \prod_{i \in N} u_i$. Like the utilitarian CUF, this form of measuring social welfare rewards increases in individual utility at all levels, but more so for the weaker agents. For instance, the vectors (1, 6, 5) and (4, 4, 4) have the same utilitarian social welfare, but the latter has a higher Nash product (and intuitively is the fairer solution of the two). For the special case of just two agents, the Nash product is discussed in more detail in the chapter on *Negotiation and Bargaining* in this volume.

Any CUF gives rise to a *social welfare ordering* (SWO), a transitive and complete order on the space of utility vectors (in the same way as an individual utility function induces a preference relation). We can also define SWOs directly. The most important example in this respect is the *leximin ordering*. For the following definition, suppose that all utility vectors are ordered, i.e., $u_1 \leq u_2 \leq \dots \leq u_n$. Under the leximin ordering, (u_1, \dots, u_n) is socially preferred to (v_1, \dots, v_n) if and

only if there exists a $k \leq n$ such that $u_i = v_i$ for all $i < k$ and $u_k > v_k$. This is a refinement of the idea underlying the egalitarian CUF. Under the leximin ordering, we first try to optimize the well-being of the worst-off agent. Once our options in this respect have been exhausted, we try to optimize the situation for the second worst-off agent, and so forth.

SWOs have been studied using the axiomatic method in a similar manner as SWFs and SCFs. Let us briefly review three examples of axioms considered in this area.

- An SWO \succeq is *zero independent* if $u \succeq v$ entails $(u + w) \succeq (v + w)$ for any $w \in \mathbb{R}^n$. That is, according to this axiom, social judgments should not change if some of the agents change their individual “zero point”. Zero independence is the central axiom in a characterization of the SWOs induced by the utilitarian CUF [83, 165].
- An SWO \succeq is *independent of the common utility pace* if $u \succeq v$ entails $(g(u_1), \dots, g(u_n)) \succeq (g(v_1), \dots, g(v_n))$ for any increasing bijection $g : \mathbb{R} \rightarrow \mathbb{R}$. You might think of g as a function that maps gross to net income. Then the axiom says that we want to be able to make social judgments independently from the details of g (modeling the taxation laws), as long as it never inverts the relative welfare of two individuals. The utilitarian SWO fails this axiom, but the egalitarian SWO does satisfy it.
- An SWO \succeq satisfies the *Pigou-Dalton principle* if $u \succeq v$ whenever u can be obtained from v by changing the individual utilities of only two agents in such a way that their mean stays the same and their difference reduces. The Pigou-Dalton principle plays a central role in the axiomatic characterization of the leximin ordering [165].

For an excellent introduction to the axiomatics of welfare economics, providing much more detail than what is possible here, we refer to the book of Moulin [165]. Broadly speaking, the additional information carried by a valuation function (on top of its ordinal content, i.e., on top of the kind of information used in voting theory), avoids some of the impossibilities encountered in the ordinal framework. For instance, if we enrich our framework with a monetary component and stipulate that each agent’s utility can be expressed as the sum of that agent’s money and his valuation for his goods (so-called *quasilinear preferences*), then we can define strategyproof mechanisms that are not dictatorial. Examples are the mechanism used in the well-known Vickrey auction and its generalizations (see the chapter on *Mechanism Design and Auctions* in this volume).

Another important fairness criterion is *envy-freeness*. An allocation A of goods is envy-free if no agent would rather obtain the bundle allocated to one of the other agents: $v_i(A(i)) \geq v_i(A(j))$ for any two agents i and j , with $A(i)$ and $A(j)$ denoting the bundles of goods allocated to i and j , respectively. Note that this concept cannot be modeled in terms of a CUF or an SWO. If we insist on allocating all goods, then an envy-free allocation will not always exist. A simple example is the case of two agents and one item that is desired by both of them: in this case, neither of the two complete allocations will be envy-free. When no envy-free allocation is possible, then we might want to aim for an allocation that minimizes the *degree of envy*. A variety of definitions for the degree of envy of an allocation have been proposed in the literature, such as counting the number of agents experiencing some form of envy or counting the pairs of agents where the first agent envies the second [155, 55, 154].

A new application in multiagent systems may very well call for a new fairness or efficiency criterion. However, any new idea of this kind should always be clearly positioned with respect to the existing standard criteria, which are well motivated philosophically and deeply understood mathematically.

5.3 Computing Fair and Efficient Allocations

Once we have settled on a language for modeling the valuation functions of individual agents and on an efficiency or fairness criterion we want to apply, the question arises of how to compute an optimal allocation of goods. Algorithmic methods that have been used in this field include linear and integer programming, heuristic-guided search, and constraint programming. Rather than discussing specific algorithms here, let us focus on the computational complexity of the combinatorial optimization problems such an algorithm would have to tackle.

First, suppose we want to compute an allocation with maximal utilitarian social welfare. In the combinatorial auction literature, this problem is generally known as the “winner determination problem” and it has received a large amount of attention there [82]. The goal is to maximize the sum of the valuations of the individual agents. As complexity theory is more neatly applied to decision problems, let us consider the corresponding decision problem:

We are given a profile of valuation functions (v_1, \dots, v_n) , one for each agent, and a number K and ask whether there exists an allocation $A : N \rightarrow 2^G$ mapping agents to bundles of goods, with $A(1) \cup \dots \cup A(n) = G$ and $A(i) \cap A(j) = \emptyset$ for any two agents i and j , such that

$f_u(v_1(A(1)), \dots, v_n(A(n))) \geq K$, i.e., such that the utilitarian social welfare of A is at least K .

This problem turns out to be NP-complete for any of the preference representation languages discussed [54, 82]. To show that it is in NP is usually easy. The fact that it is NP-hard was first observed by Rothkopf et al. [187], in the context of what is now known as the OR-language. The proof proceeds via a reduction from a standard NP-hard problem known as SET PACKING [121]: given a collection of finite sets \mathcal{C} and a number K , is there a collection $\mathcal{C}' \subseteq \mathcal{C}$ of pairwise disjoint sets such that $|\mathcal{C}'| \geq K$? Now consider any preference representation language that allows us to specify for each agent i one bundle B_i such that $v_i(B) = 1$ if $B = B_i$ (or possibly $B \supseteq B_i$) and $v_i(B) = 0$ otherwise (weighted goals, the OR-language, and the XOR-language can all do this). Then we have a one-to-one correspondence between finding an allocation with a utilitarian social welfare of at least K and finding at least K non-overlapping bundles B_i . Hence, welfare optimization must be at least as hard as SET PACKING. (The problem has also been shown to be NP-hard to approximate [190, 233].)

Let us now briefly go over some related complexity results, but with less attention to detail. Rather than stating these results precisely, we focus on some of the crucial insights they represent and cite the original sources for further details.

- Computing allocations that are optimal under the egalitarian CUF or the Nash CUF is also NP-hard [32, 180]. In case all valuation functions are *additive*, i.e., if we can always compute the value of a bundle by adding the values of the individual items in that bundle, then computing an allocation with maximal utilitarian social welfare becomes easy (simply assign each item to the agent giving it the highest value), but the same domain restriction does not render the problem polynomial when we are interested in egalitarian social welfare or the Nash product.
- An allocation A is Pareto-optimal if there is no other allocation A' that is strictly preferred by some agent and not worse for any of the others. Deciding whether a given allocation is Pareto-optimal is typically coNP-complete [94, 57, 87, 54]. The crucial difference with respect to optimizing utilitarian social welfare is that we now have to check that there is *no* other allocation that is “better” (hence the switch in complexity class).
- Computing allocations that are envy-free can be significantly more diffi-

cult.²² Bouveret and Lang [33] show that deciding whether there exists an allocation that is both Pareto-optimal and envy-free is Σ_2^P -complete when preferences are represented using weighted goals, even when each agent can only distinguish between “good” (valuation 1) and “bad” (valuation 0) bundles. When valuations are additive, then deciding whether there exists an envy-free allocation (that allocates all goods) is still NP-complete [155]. Lipton et al. [155] also discuss approximation schemes for envy-freeness.

5.4 Convergence to Fair and Efficient Allocations

The complexity results we have reviewed above apply in situations where we have complete information regarding the individual preferences of the agents and we want to compute a socially optimal allocation in a centralized manner. Of course, this is an idealized situation that we will rarely encounter in a multiagent system in practice. Besides the algorithmic challenges highlighted above, we also need to face the game-theoretical problem of ensuring that agents report their preferences truthfully (in case we consider truthfulness an important desideratum). We need to design a suitable elicitation protocol to obtain the relevant preference information from the agents.

An alternative approach, which we shall discuss next, is as follows: rather than centrally collecting all the agents’ preference information and determining an optimal allocation, we let agents locally find utility-improving deals that involve the exchange of some of the goods in their possession. We can then analyze the effects that sequences of such local trading activities have on the allocations emerging at the global level. If we give up control in this manner, we might not always be able to reach a socially optimal allocation. Instead, we now have to ask what quality guarantees we can still provide. This *distributed approach to multiagent resource allocation* requires us to fix a set of assumptions regarding the local behavior of agents. For instance, we could make the assumption that each agent is a utility-maximizer in the full game-theoretic sense. Often this will be unrealistic, given the high complexity of computing one’s optimal strategy under all circumstances in this context. Instead, let us assume that agents are *individually rational* and *myopic*. This means that we assume that an agent will agree to participate in a deal if and only if that deal increases his utility. On the other hand, he will not try to optimize his payoff in every single deal and he does not plan ahead beyond

²²One allocation that is always envy-free is the one where nobody gets anything. To prevent such trivialities, normally some efficiency requirement is added as well.

the next deal to be implemented.

Formally, a *deal* is a pair of allocations $\delta = (A, A')$ with $A \neq A'$, describing the situation before and after the exchange of goods. Note that this definition permits exchanges involving any number of agents and goods at a time. Bilateral deals, involving only two agents, or simple deals, involving only one item, are special cases. For the result we want to present in some detail here, we assume that a deal may be combined with monetary *side payments* to compensate some of the agents for a loss in utility. This can be modeled via a function $p : N \rightarrow \mathbb{R}$, mapping each agent to the amount he has to pay (or receive, in case $p(i)$ is negative), satisfying $p(1) + \dots + p(n) = 0$, i.e., the sum of all payments made must equal the sum of all payments received. A deal $\delta = (A, A')$ is *individually rational* if there exists a payment function p such that $v_i(A'(i)) - v_i(A(i)) > p(i)$ for all agents $i \in N$, with the possible exception of $p(i) = 0$ in case $A(i) = A'(i)$. That is, a deal is individually rational if payments can be arranged in such a way that for each agent involved in the deal his gain in valuation exceeds the payment he has to make (or his loss in valuation is trumped by the money he receives). We shall assume that every deal made is individually rational in this sense. Note that we do not *force* agents to make deals under these conditions; we simply assume that any deal that is implemented is (at least) individually rational.

Now, by a rather surprising result due to Sandholm [189], any sequence of individually rational deals must converge to an allocation with maximal utilitarian social welfare. That is, provided all agents are individually rational and continue to make individually rational deals as long as such deals exist, we can be certain that the resulting sequence of deals must be finite and that the final allocation reached must be socially optimal in sense of maximizing utilitarian social welfare. For a detailed discussion and a full proof of this result we refer to the work of En-driss et al. [104]. The crucial step in the proof is a lemma that shows that, in fact, a deal is individually rational if and only if it increases utilitarian social welfare. Convergence then follows from the fact that the space of possible allocations is finite.

How useful is this convergence result in practice? Of course, the complexity results discussed in Section 5.3 did not just go away: finding an allocation that maximizes utilitarian social welfare is still NP-hard. Indeed, to decide whether it is possible to implement yet another individually rational deal, our agents do have to solve an NP-hard problem (in practice, most of the time they might find it easy to identify a good deal, but in the worst case this can be very hard). Also the *structural complexity* of the deals required (in the worst case) is very high. Indeed, if our agents use a negotiation protocol that excludes deals involving a certain num-

ber of agents or goods, then convergence cannot be guaranteed any longer [104]. On the other hand, a simple positive result shows that, if all valuation functions are *additive*, then we can get away with a protocol that only allows agents to make deals regarding the reallocation of one item at a time. Unfortunately, this is the best result we can hope for along these lines: for no strictly larger class of valuation functions will a simple protocol of one-item-at-a-time deals still suffice to guarantee convergence [60].

Similar results are also available for other fairness and efficiency criteria, such as Pareto-optimality [104], egalitarian social welfare [104], and envy-freeness [55]. Most of the work in the field is of a theoretical nature, but the convergence problem has also been studied using agent-based simulations [4, 50].

Some of the results in the literature are based on the same notion of myopic individual rationality used here and others rely on other *rationality assumptions*. In fact, there are two natural perspectives regarding this point. First, we might start by postulating reasonable assumptions regarding the rational behavior of individual agents and then explore what convergence results can be proven. Second, we might start with a convergence property we would like to be able to guarantee, and then *design* appropriate rationality assumptions that will allow us to prove the corresponding theorem. That is, we may think of a multiagent system as, first, a system of self-interested agents we cannot control (but about which we can make certain assumptions) or, second, as a system of agents the behavior of which we can design and program ourselves, as a tool for distributed problem solving. Which perspective is appropriate depends on the application at hand.

Finally, the distributed approach to multiagent resource allocation also gives rise to new questions regarding *computational complexity*. For instance, we might ask how hard it is to decide whether a given profile of valuation functions and a given initial allocation admit a path consisting only of individually rational deals involving the exchange of a single item each to a socially optimal allocation. Questions of this kind have been studied in depth by Dunne and colleagues [94, 93].

6 Conclusion

This chapter has been an introduction to classical social choice theory and an exposition of some of the most important research trends in computational social choice. We have argued in the beginning that social choice theory, the mathematical theory of collective decision making, has a natural role to play when we

think about the foundations of multiagent systems. As we are concluding the chapter, we would like to relativize this statement somewhat: it is true that many decision problems in multiagent systems are naturally modeled as problems of social choice, but it is also true that many of the problems that one is likely to encounter in practice will not fit the template provided by the classical formal frameworks introduced here exactly, or will have additional structure that can be exploited. More research is required to improve our understanding of best practices for adapting the elegant mathematical tools that computational social choice can provide to the problems encountered by practitioners developing real multiagent systems. We hope that readers of this chapter will feel well equipped to participate in this undertaking.

Let us conclude with a brief review of additional topics in computational social choice, which we have not been able to cover in depth here, as well as with a few pointers to further reading.

6.1 Additional Topics

In terms of social choice *settings*, we have covered preference aggregation, voting, and fair division. Another important area of social choice theory is *matching*, which addresses the problem of how to pair up the elements of two groups that have preferences over each other (e.g., men and women, doctors and hospitals, or kidney donors and patients). Matching theory is particularly notable for its many successful applications. An excellent introduction to the field is the seminal monograph by Roth and Sotomayor [184]. Matching can be seen as a special case of coalition formation where agents have preferences over the various possible partitions of the set of agents (see the chapter on *Computational Coalition Formation* in this volume).

Preferences are not the only individual characteristics that the members of a group might want to aggregate. Other examples include beliefs and judgments. In both cases there exists a small but significant technical literature in which beliefs and judgments, respectively, are modeled as sets of formulas in propositional logic that need to be aggregated. The work of Konieczny and Pino Pérez [143] is a good starting point for reading about *belief merging* and List and Puppe [156] survey the literature on *judgment aggregation*. While belief merging grew out of the literature on belief revision in artificial intelligence and computational logic and always had a computational flavor to it, judgment aggregation initially developed in the political philosophy and the philosophical logic literature and computational aspects did not get investigated until very recently [105].

Throughout the chapter, we have occasionally alluded to connections to *mechanism design*, a topic at the interface of social choice and game theory (see also the chapter on *Mechanism Design and Auctions* in this volume). On the mechanism design side, there has been interest in designing voting rules that are *false-name-proof* [227], that is, robust to a single voter participating under multiple identities. While this is not an inherently computational topic, it is motivated by applications such as elections that take place on the Internet. The design of such rules has been studied both in general [62] and under single-peaked preferences [208]. Unfortunately, the results are rather negative here. To address this, other work has extended the model, for example by making it costly to obtain additional identities [212] or by using social network structure to identify “suspect” identities [77]. An overview of work on false-name-proofness is given by Conitzer and Yokoo [72]. Another exciting new direction in the intersection of computational social choice and mechanism design is that of *approximate mechanism design without money* [178], where the goal is to obtain formal approximation ratios under the constraint of strategyproofness, without using payments.

In terms of *techniques*, we have focused on the axiomatic method, on algorithms, and on computational complexity. We have also discussed the use of tools from knowledge representation (for the representation of preferences in combinatorial domains). A further important research trend in computational social choice has considered *communication requirements* in social choice. This includes topics such as the amount of information that voters have to supply before we can compute the winner of an election [69, 196], the most efficient form of storing an intermediate election result that will permit us to compute the winner once the remaining ballots have been received [59, 219], whether voters can jointly compute the outcome of a voting rule while preserving the privacy of their individual preferences [44], and the number of deals that agents have to forge before a socially optimal allocation of goods will be found [103].

Another technique we have not discussed concerns the use of tools developed in *automated reasoning* to verify properties of social choice mechanisms and to confirm or discover theorems within social choice theory. Examples in this line of work include the verification of proofs of classical theorems in social choice theory in higher-order theorem provers [168], a fully automated proof of Arrow’s Theorem for the special case of three alternatives [205], and the automated discovery of theorems pertaining to the problem of ranking sets of objects [123].

6.2 Further Reading

There are a number of excellent textbooks on classical social choice theory that are highly recommended for further reading. General texts include those by Austen-Smith and Banks [7] and by Gaertner [120]. Taylor [206] specifically focuses on the manipulation problem in voting. Moulin [165] covers not only preference aggregation and voting, but also the axiomatic foundations of welfare economics (i.e., fair division) and cooperative game theory. Two highly recommended surveys are those of Plott [173] and Sen [200].

The area of computational social choice (or certain parts thereof) has been surveyed by several authors. Chevaleyre et al. [56] provide a broad overview of the field as a whole. The literature on using computational complexity as a barrier against manipulation in voting is reviewed by Faliszewski et al. [114] and Faliszewski and Procaccia [107]; Faliszewski et al. [110] also discuss the complexity of winner determination and control problems in depth. Chevaleyre et al. [58] give an introduction to social choice in combinatorial domains. The survey on multiagent resource allocation by Chevaleyre et al. [54] covers the basics of fair division and also discusses connections to other areas relevant to multiagent systems, particularly combinatorial auctions. Conitzer [64, 65] compares research directions across mechanism design, combinatorial auctions, and voting. Endriss [101] gives concise proofs of classical results such as Arrow's Theorem and the Gibbard-Satterthwaite Theorem, and then discusses application of logic in social choice theory, e.g., in judgment aggregation and to model preferences in combinatorial domains. Rothe et al. [186] provide a book-length introduction to computational social choice (in German), covering topics in voting, judgment aggregation, and fair division, and focusing particularly on complexity questions. Finally, the short monograph of Rossi et al. [183] on preference handling includes extensive discussions of voting and matching from the point of view of computational social choice.

7 Exercises

1. **Level 1** A utility function $u : U \rightarrow \mathbb{R}$ is said to *represent* a preference relation on U if, for all $a, b \in U$, $u(a) \geq u(b)$ if and only if $a \succeq b$. Show that, when U is finite, a preference relation can be represented by a utility function if and only if it is transitive and complete.
2. An SWF f is *non-imposing* if for every preference relation \succeq there exists

a profile $R = (\succsim_1, \dots, \succsim_n)$ such that $f(R) = \succsim$. The purpose of this exercise is to investigate what happens to Arrow's Theorem when we replace the axiom of Pareto-optimality by the axiom of non-imposition.

- (a) **Level 1** Show that Pareto-optimality is strictly stronger than non-imposition. That is, show that every Pareto-optimal SWF is non-imposing and that there exists a non-imposing SWF that is not Pareto-optimal.
 - (b) **Level 2** Show that Arrow's Theorem ceases to hold when we replace Pareto-optimality by non-imposition. That is, show that there exists a SWF that satisfies IIA and that is both non-imposing and non-dictatorial.
3. **Level 2** Prove that the four conditions in Theorem 2 are logically independent by providing, for each of the conditions, an SCF that violates this property but satisfies the other three.
 4. **Level 2** Show that every Copeland winner lies in the uncovered set and hence reaches every other alternative on a majority rule path of length at most two (assuming an odd number of voters).
 5. **Level 1** Consider the following preference profile for 100 voters (due to Michel Balinski).

33	16	3	8	18	22
<i>a</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>d</i>	<i>e</i>
<i>b</i>	<i>d</i>	<i>d</i>	<i>e</i>	<i>e</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>c</i>	<i>b</i>
<i>d</i>	<i>e</i>	<i>a</i>	<i>d</i>	<i>b</i>	<i>d</i>
<i>e</i>	<i>a</i>	<i>e</i>	<i>a</i>	<i>a</i>	<i>a</i>

Determine the winners according to plurality, Borda's rule, Copeland's rule, STV, and plurality with runoff (which yields the winner of a pairwise comparison between the two alternatives with the highest plurality score).

6. **Level 2** Give a polynomial-time algorithm that, for a given preference profile, decides whether an alternative will win under *all* scoring rules.
7. **Level 3** A *Condorcet loser* is an alternative that loses against every other alternative in pairwise majority comparisons. Check which of the following

voting rules may choose a Condorcet loser: Borda's rule, Nanson's rule, Young's rule, maximin. Prove your answers.

8. **Level 3** An SCF is *monotonic* if a winning alternative will still win after it has been raised in one or more of the individual preference orderings (leaving everything else unchanged). Check which of the SCFs and voting rules mentioned in this chapter satisfy monotonicity and which satisfy Pareto-optimality. Prove your answers.
9. **Level 2** Assume there is an odd number of voters and consider a restricted domain of preferences that always admits a Condorcet winner. Show that the voting rule that always yields the Condorcet winner is strategyproof.
10. **Level 2** Assume there is an odd number of voters, and rank the alternatives by their Copeland scores. Prove that there are no cycles in the pairwise majority relation if and only if no two alternatives are tied in this Copeland ranking.
11. **Level 2** Recall the definition of single-peakedness. Similarly, a preference profile R is *single-caved* if for every $x, y, z \in U$, it holds that if $(x < y < z)$ or $(z < y < x)$, then $y >_i x$ implies $z >_i y$ for every $i \in N$. Prove or disprove the following statements.
 - (a) Every preference profile for two voters and three alternatives is single-peaked.
 - (b) Every preference profile for two voters and more than three alternatives is single-peaked.
 - (c) Every single-peaked preference profile is single-peaked with respect to the linear order given by the preferences of one of the voters.
 - (d) Plurality and Condorcet winners coincide for single-peaked preferences.
 - (e) Plurality and Condorcet winners coincide for single-caved preferences.
 - (f) Borda and Condorcet winners coincide for single-peaked preferences.
12. **Level 4** We have seen that any non-dictatorial voting rule can be manipulated when we want that rule to operate on all possible preference profiles. We have also seen that this problem can be avoided when we restrict the

domain of possible profiles appropriately, e.g., to single-peaked profiles. What we have not discussed is the *frequency* of manipulability: how often will we encounter a profile in which a voter has an incentive to manipulate? One way of studying this problem is by means of simulations: generate a large number of profiles and check for which proportion of them the problem under consideration (here, manipulability) occurs. The standard method for generating profiles is to make the *impartial culture assumption*, under which every logically possible preference order has the same probability of occurring. For instance, if there are 3 alternatives, then there are $3! = 6$ possible (strict) preference orders, so each preference order should have probability $\frac{1}{6}$ to be a given voter's preference.

- (a) Write a program to analyze the frequency of manipulability of some of the voting rules introduced in this chapter under the impartial culture assumption.
- (b) While it is considered a useful base line, the impartial culture assumption has also been severely criticized for being too simplistic. Indeed, real electorates, be it in politics or multiagent systems, are unlikely to be impartial cultures. Can you think of better methods for generating data to test the frequency of interesting phenomena in social choice theory?

A good starting point for further reading on generating data for studying the frequency of social choice phenomena is the book of Regenwetter et al. [181]. There has also been a significant amount of theoretical work on the frequency of manipulability [recent contributions include, e.g., 8, 203, 177, 218].

13. **Level 2** Show that for each of the following voting rules the manipulation problem (with a single manipulator) can be solved in polynomial time by providing a suitable algorithm: the plurality rule, Borda's rule, Copeland's rule. Argue why your algorithms are correct and analyze their runtime in terms of the number of voters and alternatives.
14. For some voting rules, it is possible to significantly reduce the amount of information that the voters need to communicate by having the communication take place in rounds. A natural example is the STV rule (also known as instant runoff voting). Instead of having each agent communicate an entire ranking of all the alternatives at the outset, we can simply have the agents

communicate their first-ranked alternatives; based on that, we can determine which alternative gets eliminated first; then, the agents who had ranked that alternative first communicate their next-most preferred alternative; etc. In effect, this is removing the “instant” from “instant runoff voting”!

- (a) **Level 2** When there are n voters and m alternatives, how many bits of communication does this protocol require in the worst case? Hints:
- If there are i alternatives left, how many bits does an agent need to communicate to indicate its most-preferred one among them?
 - If there are i alternatives left and we remove the one with the fewest votes, what is an upper bound on how many agents need to indicate a new most-preferred alternative among the $i - 1$ remaining ones?
- (b) **Level 4** Using tools from communication complexity [144], a lower bound of $\Omega(n \log m)$ bits of information in the worst case has been shown to hold for *any* communication protocol for the STV rule [69]. This leaves a gap with the result from (a). Can you close the gap, either by giving a better protocol or a better lower bound?
15. **Level 2** A *weak Condorcet winner* is an alternative that wins or draws against any other alternative in pairwise contests. Just like a (normal) Condorcet winner, a weak Condorcet winner need not exist for all preference profiles. Unlike a Condorcet winner, when it does exist, a weak Condorcet winner need not be unique. In the context of voting in combinatorial domains, show that when voters model their preferences using the language of prioritized goals and each voter only specifies a single goal, then there must always be a weak Condorcet winner.
16. **Level 1** In the context of measuring the fairness and efficiency of allocations of goods, check which of the following statements are true. Give either a proof (in the affirmative case) or a counterexample (otherwise).
- (a) Any allocation with maximal utilitarian social welfare is Pareto-optimal.
 - (b) No allocation can maximize both utilitarian and egalitarian social welfare.
 - (c) Any allocation that is optimal with respect to the leximin ordering is both Pareto-optimal and maximizes egalitarian social welfare.

- (d) The Nash SWO is zero independent.
 - (e) The Nash SWO is independent of the common utility pace.
 - (f) The egalitarian SWO respects the Pigou-Dalton transfer principle.
17. **Level 2** The *elitist* CUF is defined via $f_{el}(u_1, \dots, u_n) := \max\{u_i \mid i \in N\}$, i.e., social welfare is equated with the individual utility of the agent that is currently best off. This CUF clearly contradicts our intuitions about fairness, but it might be just the right efficiency measure for some applications, e.g., in a multiagent system where we only care about at least one agent achieving his goal. What is the computational complexity of (the decision variant of) the problem of finding an allocation of indivisible goods (without money) that maximizes elitist social welfare?
- (a) First state your answer (and your proof) with respect to the explicit form of representing valuation functions (where the size of the representation of a function is proportional to the number of bundles to which it assigns a non-zero value).
 - (b) Then repeat the same exercise, this time assuming that valuation functions are expressed using the language of weighted goals (without restrictions to the types of formulas used). Hint: You might expect that the complexity will increase, because now the input will be represented more compactly (on the other hand, as discussed in Section 5.3, there was no such increase in complexity for the utilitarian CUF).

Note that both of these languages can express valuation functions that need not be monotonic (that is, simply giving all the items to one agent will usually not yield an allocation with maximal elitist social welfare).

18. **Level 4** Consider a fair division problem with an odd number of agents. Under the *median-rank dictator* CUF the social welfare of an allocation is equal to the utility of the middle-most agent: $f_{mrd}(u_1, \dots, u_n) := u_{i^*}$, where i^* is defined as the (not necessarily unique) agent for which $|\{i \in N \mid u_i \leq u_{i^*}\}| = |\{i \in N \mid u_i \geq u_{i^*}\}|$. This is an attractive form of measuring social welfare: it associates social welfare with the individual utility of a representative agent, while being less influenced by extreme outliers than, for instance, the utilitarian CUF. At the time of writing, most of the problems discussed in the section on fair division have not yet been investigated for the median-rank dictator CUF.

- (a) What is the computational complexity of computing an allocation that is optimal under the median-rank dictator CUF? Consider this question for different forms of representing individual valuation functions, such as the explicit form, weighted propositional formulas, or the OR/XOR family of bidding languages used in combinatorial auctions.
- (b) Design and implement an algorithm for computing an optimal allocation under the median-rank dictator CUF, for a preference representation language of your choice. You may find it useful to consult the literature on efficient algorithms for the winner determination problem in combinatorial auctions to get ideas on how to approach this task.
- (c) Can you devise a notion of rationality (replacing myopic individual rationality as defined in this chapter) so that distributed negotiation will guarantee convergence to an optimal allocation under the median-rank dictator CUF? Are there suitable domain restrictions (limiting the diversity of valuation functions that agents may hold) that will ensure convergence even when negotiation is limited to structurally simple deals (such as deals involving at most two agents at a time)?

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