The Distribution of Optimal Strategies in Symmetric Zero-sum Games

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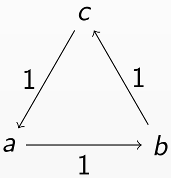
GAMES 2016

An Example



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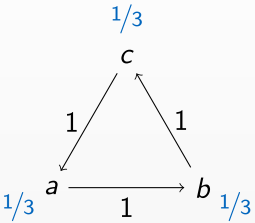

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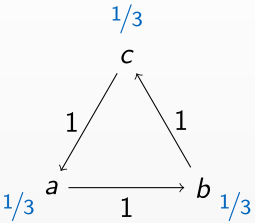

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$$p^{*T} M \geq 0$$

An Example

$$M = \begin{pmatrix} 0 & m_{1,2} & \dots & m_{1,n} \\ -m_{1,2} & 0 & \ddots & \vdots \\ \vdots & \ddots & 0 & m_{n-1,n} \\ -m_{1,n} & \dots & -m_{n-1,n} & 0 \end{pmatrix}$$

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- p^* an *optimal (maximin) strategy* of M,
- $p_+ = \{i \in A \colon p_i > 0\}$ the *support* of a strategy p, and
- $\mathcal{X}$ a *probability distribution* over symmetric zero-sum games.

Regularity

$$\begin{pmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ 2 & 1 & -1 & 0 \end{pmatrix}^{*}$$

*Le Breton [2005]

Regularity

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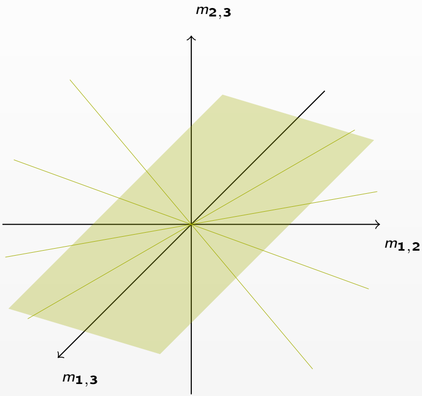

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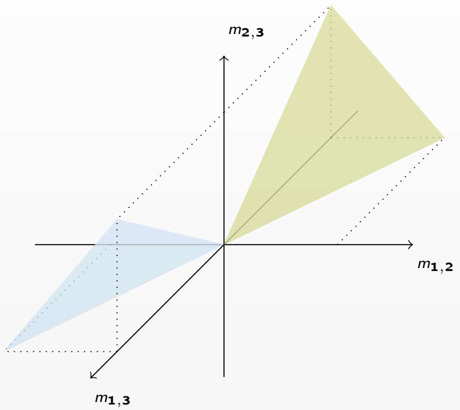
Symmetry



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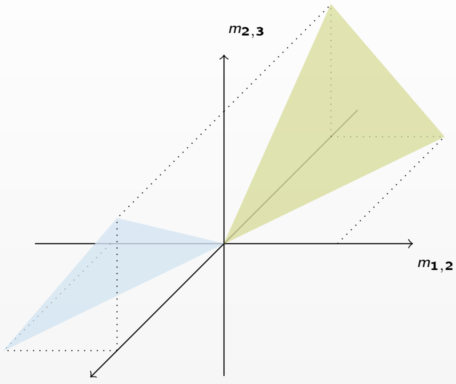
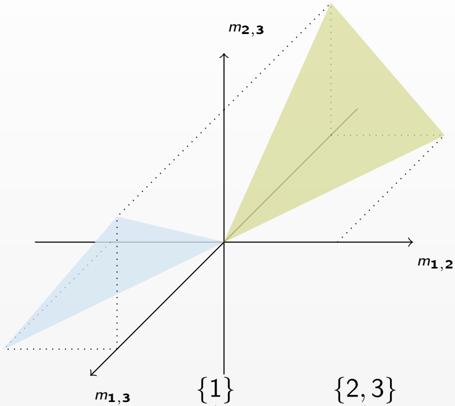


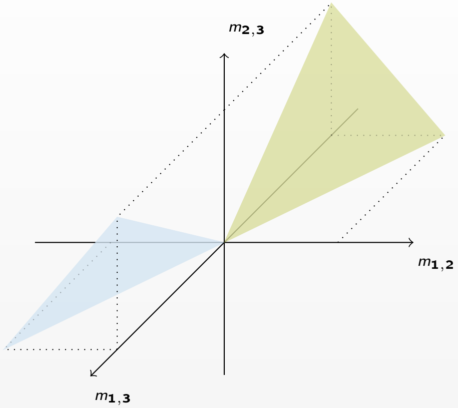
$$\phi_S(M) = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\quad}^{S} & \overbrace{\qquad\quad}^{A \setminus S} \end{array} \\ \left(\begin{array}{c|c} m_{ij} & -m_{ij} \\ \hline -m_{ij} & m_{ij} \end{array}\right) \end{array}$$

Symmetry



$$\phi_{\{1\}}(M) = \begin{array}{c} \begin{array}{cc} \{1\} & \{2,3\} \end{array} \\ \left(\begin{array}{c|cc} 0 & -m_{12} & -m_{13} \\ \hline -m_{21} & 0 & m_{23} \\ -m_{31} & m_{32} & 0 \end{array}\right) \end{array}$$

Symmetry



$$P_{\mathcal{X}}(\mathcal{M}) = P_{\mathcal{X}}(\phi_S(\mathcal{M}))$$

The Result

Let $\mathcal{X}$ be a regular and symmetric probability distribution. Then, for every $S \subseteq A$, the probability that $M \sim \mathcal{X}$ has an *optimal strategy* with *support* S is

$$\begin{cases} 0 & \text{if } |S| \text{ is even, and} \\ 2^{-(n-1)} & \text{if } |S| \text{ is odd.} \end{cases}$$

Applications

- Tournament games [Fisher and Reeves, 1995]

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- Symmetric absolutely continuous distributions