Proving the Incompatibility of Efficiency and Strategyproofness via SMT Solving

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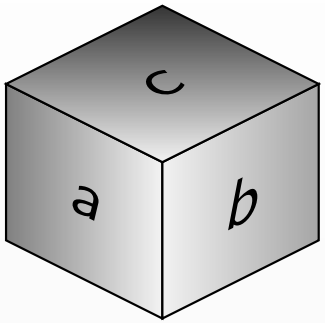
(with **Felix Brandt** and **Christian Geist**)

COMSOC 2016

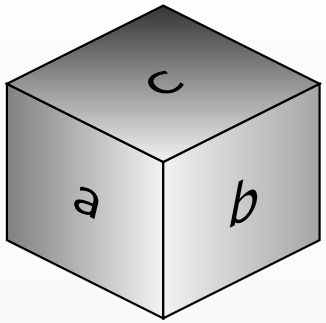
a	b	c	a	c
b	a	a	c	b
c	c	b	b	a

a	b	c	a	c
b	a	a	c	b
c	c	b	b	a

$f(\cdot)$



$1/3\,a + 1/3\,b + 1/3\,c$



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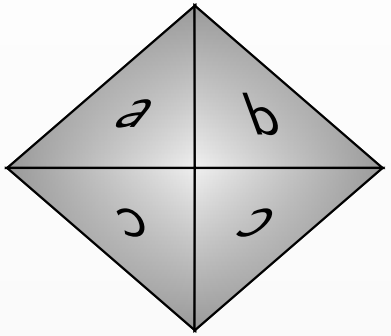
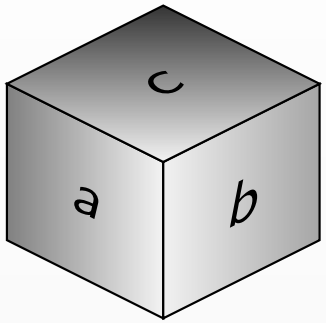


$1/4\,a + 1/4\,b + 1/2\,c$

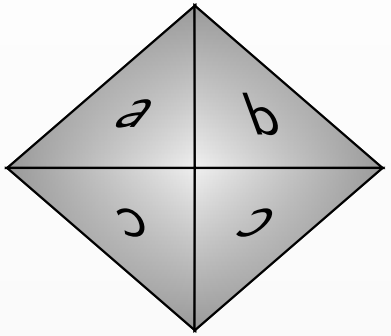


u

a

b

c



$1/3\,a + 1/3\,b + 1/3\,c$

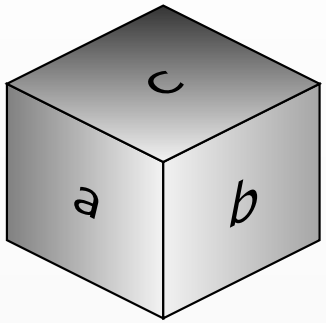
$1/3\,3 + 1/3\,1 + 1/3\,0 = 4/3$

$1/4\,a + 1/4\,b + 1/2\,c$

$1/4\,3 + 1/4\,1 + 1/2\,0 = 1$

utility function:

$$u = (3, 1, 0)$$



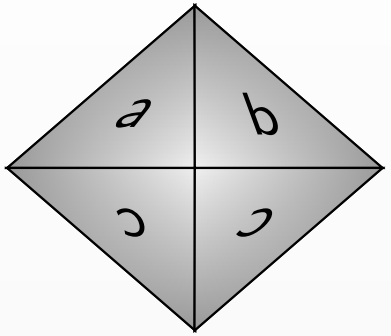
$1/3\,a + 1/3\,b + 1/3\,c$

$\succ$

a

b

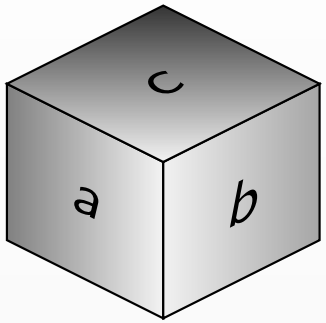
c



$1/4\,a + 1/4\,b + 1/2\,c$

$\forall$utility functions:

$p \succsim q$ iff $\sum p(x)u(x) \geq \sum q(x)u(x)$ for all *consistent* u

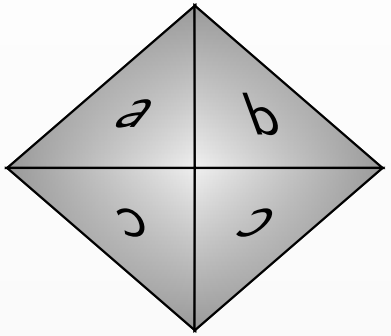


$\succ$

a

b

c



$1/3\,a + 1/3\,b + 1/3\,c$

$1/4\,a + 1/4\,b + 1/2\,c$

$\forall$utility functions $\Leftrightarrow$ stochastic dominance:

$p \succsim q$ iff $\sum p(x)u(x) \geq \sum q(x)u(x)$ for all *consistent* u

$\Leftrightarrow p \succsim q$ iff $\sum_{y \succsim x} p(y) \geq \sum_{y \succsim x} q(y)$ for all x.

Efficiency

1	2	3	4
a, c	b, d	a, d	b, c
b	a	b	a
d	c	c	d

$$1/4\, a + 1/4\, b + 1/4\, c + 1/4\, d$$

Efficiency

	1	2	3	4	
$1/2$	a, c	b, d	a, d	b, c	$1/2$
$1/4$	b	a	b	a	$1/2$
$1/4$	d	c	c	d	0

$$1/2\, a + 1/2\, b$$

$$\succ_i$$

$$1/4\, a + 1/4\, b + 1/4\, c + 1/4\, d$$

Efficiency

	1	2	3	4	
$1/2$	a, c	b, d	a, d	b, c	$1/2$
$1/4$	b	a	b	a	$1/2$
$1/4$	d	c	c	d	0

$$1/2\, a + 1/2\, b$$

$$\succ_i$$

$$1/4\, a + 1/4\, b + 1/4\, c + 1/4\, d$$

The outcome is inefficient!

Strategyproofness

1	2
a	c
b	d
c	a
d	b

$$1/2\, a + 1/2\, c$$

Strategyproofness

1	1	2
a	a	c
b	b	d
d	c	a
c	d	b

$$a$$

$$\succ_1$$

$$1/2\, a + 1/2\, c$$

Strategyproofness

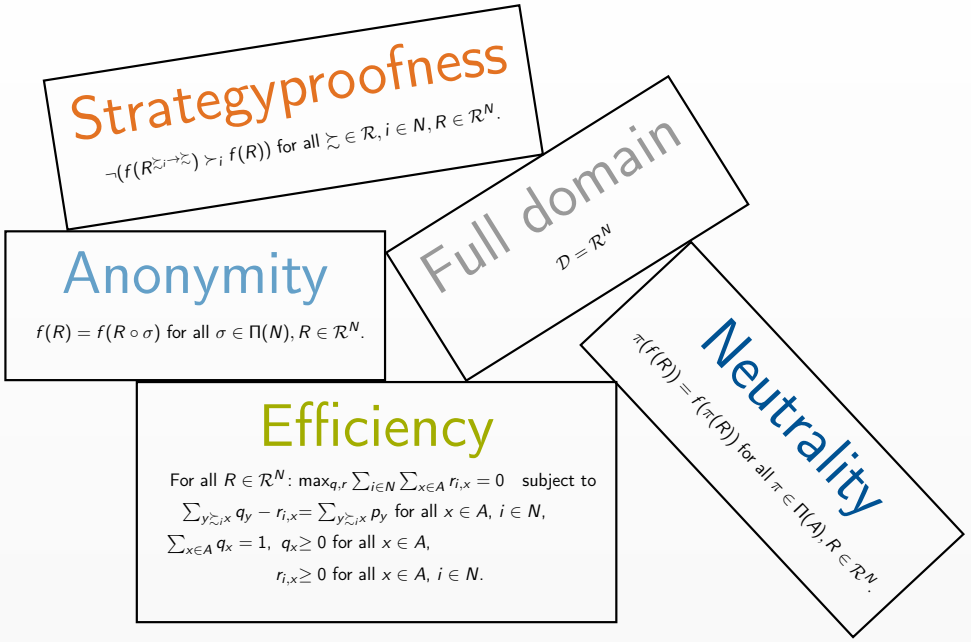
1	1	2
a	a	c
b	b	d
d	c	a
c	d	b

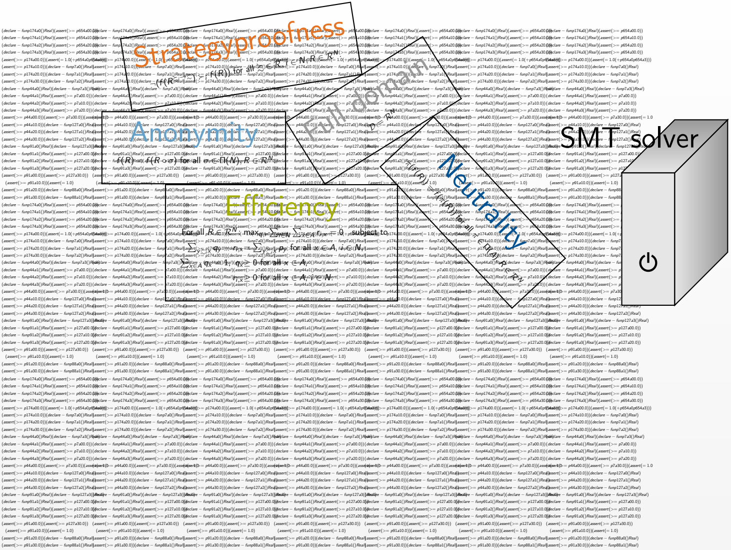
$$a$$

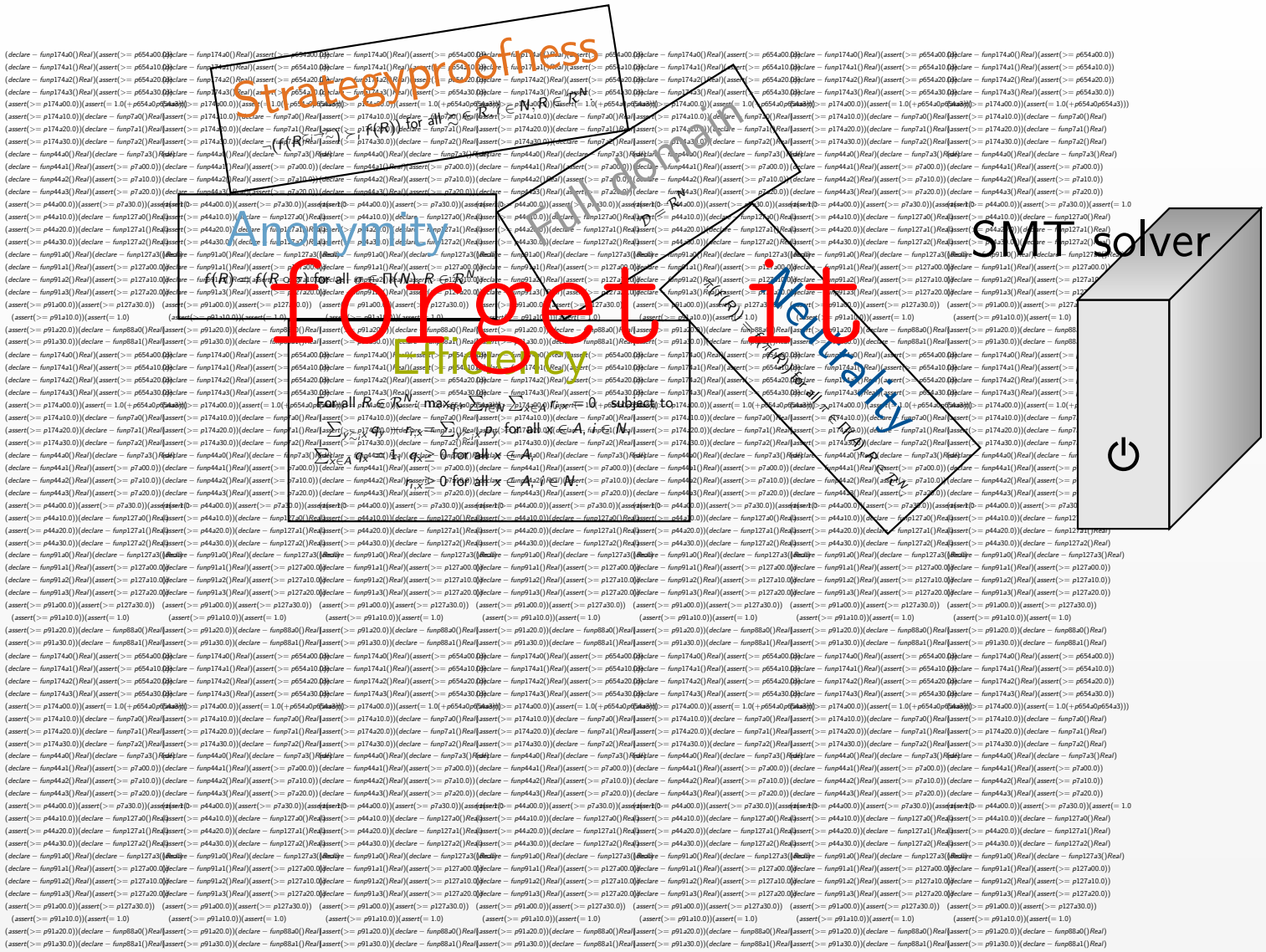
$$\succ_1$$

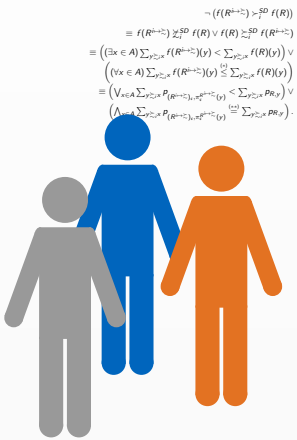
$$1/2\, a + 1/2\, c$$

Agent 1 can manipulate!


Strategyproofness
$\neg(f(R^{\succsim_i \to \succsim}) \succ_i f(R))$ for all $\succsim \in \mathcal{R}, i \in N, R \in \mathcal{R}^N$.
Full domain
$\mathcal{D} = \mathcal{R}^N$
Anonymity
$f(R) = f(R \circ \sigma)$ for all $\sigma \in \Pi(N), R \in \mathcal{R}^N$.
Neutrality
$\pi(f(R)) = f(\pi(R))$ for all $\pi \in \Pi(A), R \in \mathcal{R}^N$.
Efficiency
For all $R \in \mathcal{R}^N$: $\max_{q,r} \sum_{i \in N} \sum_{x \in A} r_{i,x} = 0$ subject to
$\sum_{y \succsim_i x} q_y - r_{i,x} = \sum_{y \succsim_i x} p_y$ for all $x \in A$, $i \in N$,
$\sum_{x \in A} q_x = 1$, $q_x \geq 0$ for all $x \in A$,
$r_{i,x} \geq 0$ for all $x \in A$, $i \in N$.


Strategyproofness
Anonymity
Full domain
Efficiency
Neutrality
SMT solver


Strategyproofness
Efficiency
Anonymity
Neutrality
Impossibility
SMT solver



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Restricted domain

$\mathcal{D} = \{R \in \mathcal{R}^N : dist(R, R^*) \leq 6\}$

Efficiency

$\bigvee_{x \in I_R} p_{R,x} = 0$ for all $R \in \mathcal{R}_c^N$.

Orbit condition

$p_{R,x} = p_{R,y}$ for all $R \in \mathcal{R}_c^N$, $O \in \mathcal{O}_R$, $x, y \in O$.

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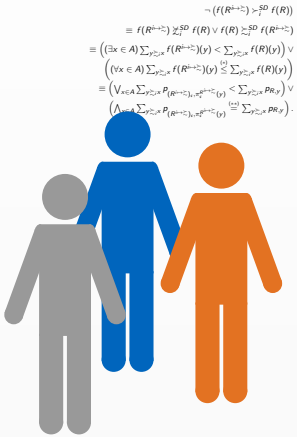

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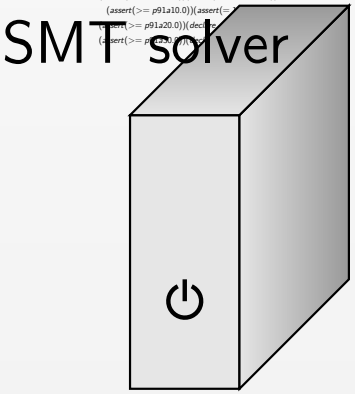

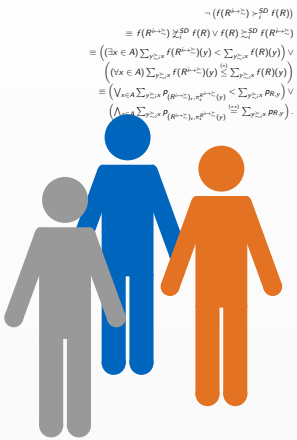

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$p_{R,x} = p_{R,y}$ for all $R \in \mathcal{R}_c^N$, $\mathcal{O} \in \mathcal{O}_R$, $x, y \in O$.

Efficiency

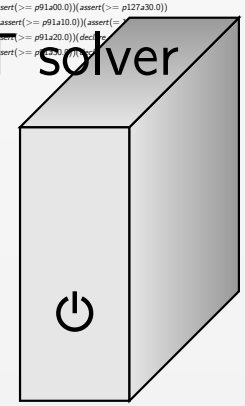
$\forall_{x \in I_R} p_{R,x} = \frac{1}{|I_R|}$ for all $R \in \mathcal{R}_c^N$.

Strategyproofness

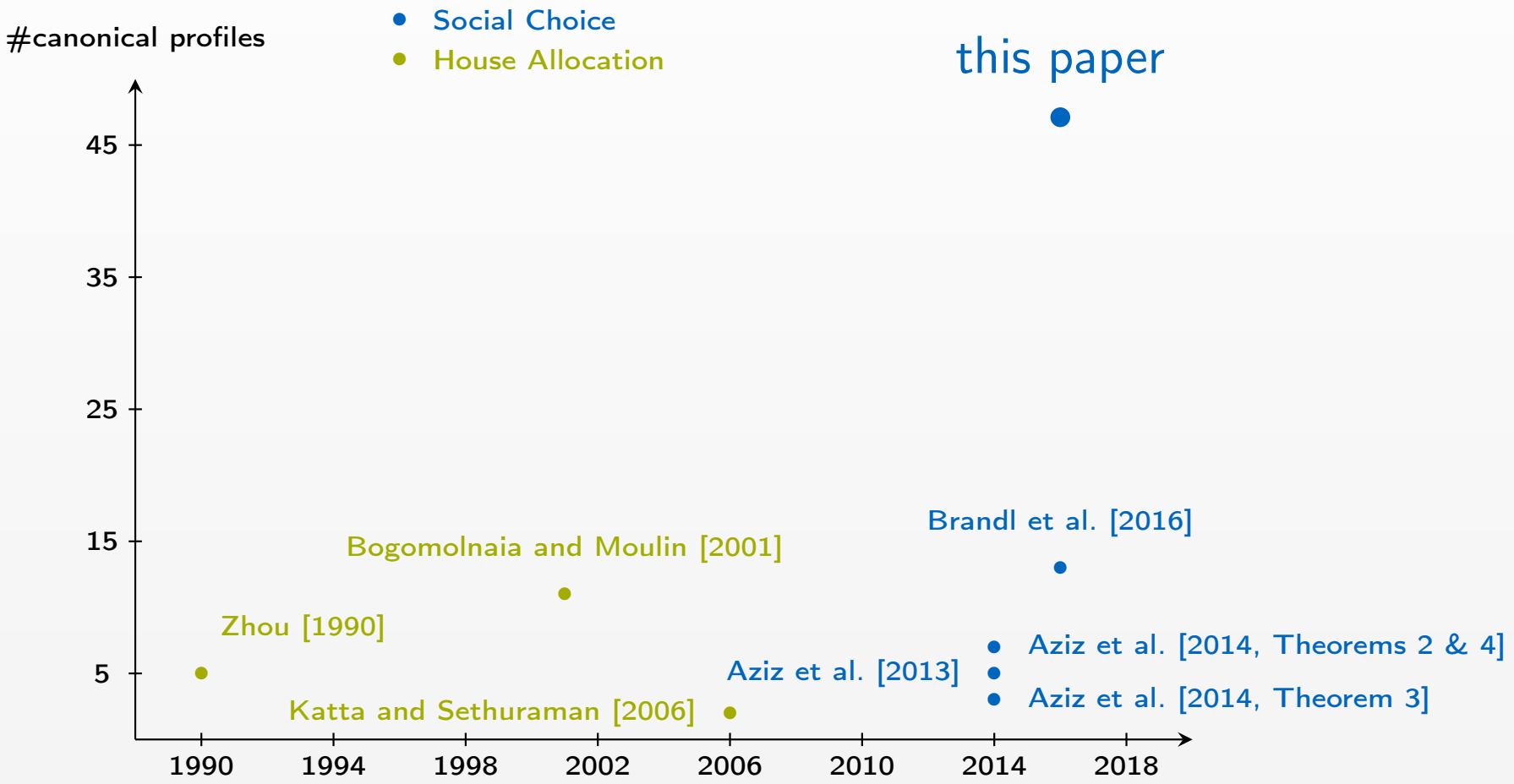
$\neg(f(R^{\succsim_i \to \succsim}) \succ_i f(R))$ for all $\succsim \in \mathcal{R}, i \in N, R \in \mathcal{R}_c^N$.

unsatisfiable

SMT solver



Proof Complexity of Related Results



Proof Complexity of Related Results