Arrovian Aggregation of Convex Preferences and Pairwise Utilitarianism

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(joint work with Felix Brandt)

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Arrovian Aggregation

Arrow's Theorem: Every social welfare function $f : \mathcal{L}^N \to \mathcal{L}$ that satisfies Pareto optimality and independence of irrelevant alternatives is dictatorial.

Weakened *collective* rationality assumptions:

- Quasi-transitivity (Gibbard, 1969) or acyclicity (Mas-Colell and Sonnenschein, 1972)

Restricted *individual* preferences:

- vNM preferences (Kalai and Schmeidler, 1977; d'Aspremont and Gevers, 2002).
- Economic public and private good domains (Kalai et al., 1979; Bordes and Le Breton, 1989, 1990).
- Positive results for single-peaked (Black, 1948) and dichotomous preferences (Inada, 1964).

Why Transitivity?

"Once considered a cornerstone of rational choice theory, the status of transitivity has been dramatically reevaluated by economists and philosophers in recent years" (Anand, 2009)

Theorem (Sonnenschein, 1971): Every continuous and convex preference relation admits a maximal element in every non-empty, compact, and convex set.

Choosing maximal elements from continuous and convex relations satisfies standard choice consistency conditions introduced by Sen (1969, 1971).

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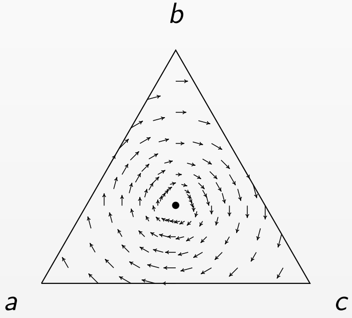

$a \succ b \succ c \succ a$

Preferences over Outcomes

- U a non-empty universal set of *alternatives*
- Δ the set of probability measures (or *outcomes*) on U with finite support
- Δ_X the set of probability measures with support in $X \subseteq U$
- $\succ$ an asymmetric *preference relation* on Δ

$\succ$ is continuous if, for all $p, q, r \in \Delta$,

$$p \succ q \succ r \text{ implies } \lambda p + (1-\lambda) r \sim q \text{ for some } \lambda \in (0,1).$$

$\succ$ is convex if, for all $p, q, r \in \Delta$ and $\lambda \in (0,1)$,

$$p \succ q \text{ and } p \succsim r \text{ imply } p \succ \lambda q + (1-\lambda) r,$$
$$q \succ p \text{ and } r \succsim p \text{ imply } \lambda q + (1-\lambda) r \succ p, \text{ and}$$
$$p \sim q \text{ and } p \sim r \text{ imply } p \sim \lambda q + (1-\lambda) r.$$

Preferences over Outcomes

$\succ$ is symmetric if, for all $p, q, r \in \Delta$ and $\lambda \in (0,1)$,

$p \succ q \succ r$, $p \succ r$, and $q \sim {}^1\!/_2\, p + {}^1\!/_2\, r$ implies
$[\lambda p + (1-\lambda) r \sim {}^1\!/_2\, p + {}^1\!/_2\, q$ iff $\lambda r + (1-\lambda) p \sim {}^1\!/_2\, r + {}^1\!/_2\, q]$.

The set of continuous, convex, and symmetric preference relations over Δ is denoted by $\mathcal{R}$.

Fishburn (1982): If $\succ \in \mathcal{R}$, then there is a unique *skew-symmetric and bilinear (SSB)* function $\phi^{\succ} \colon U \times U \to \mathbb{R}$ such that, for all $p, q \in \Delta$,

$$p \succ q \text{ iff } \phi^{\succ}(p, q) > 0.$$

SSB utility theory is more general than vNM utility theory and weighted utility theory (Chew, 1983).

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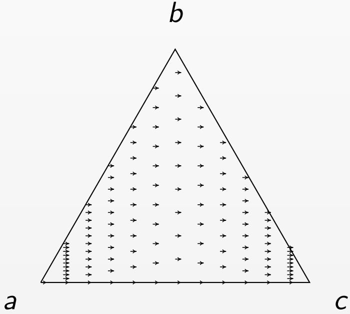

Social Welfare Functions

- $N = \{1, \dots, n\}$ a finite set of *agents*
- $\mathcal{D} \subseteq \mathcal{R}$ a *domain* of preference relations
- $f \colon \mathcal{D}^N \to \mathcal{R}$ a *social welfare function (SWF)*

f satisfies Pareto optimality if, for all $p, q \in \Delta$, $R \in \mathcal{D}^N$, and $f(R) = \succ$,

$$p \succsim_i q \text{ for all } i \in N \text{ implies } p \succsim q \text{, and}$$
$$\text{if additionally } p \succ_i q \text{ for some } i \in N \text{ then } p \succ q.$$

f satisfies Independence of irrelevant alternatives (IIA) if, for all $R, \hat{R} \in \mathcal{D}^N$ and $X \subseteq U$,

$$R|_{\Delta_X} = \hat{R}|_{\Delta_X} \text{ implies } f(R)|_{\Delta_X} = f(\hat{R})|_{\Delta_X}.$$

Characterization of the Domain

$\succ$ is based on pairwise comparisons if, for all $p, q \in \Delta$,

$$p \succ q \text{ iff } \sum_{a,b \in U\colon a \succ b} p(a)q(b) > \sum_{a,b \in U\colon b \succ a} p(a)q(b).$$

p is preferred to q if and only if p is more likely to return a more preferred alternative.

Preferences in $\mathcal{R}$ based on pairwise comparisons are characterized by the fanning-in axiom (Blavatskyy, 2006).

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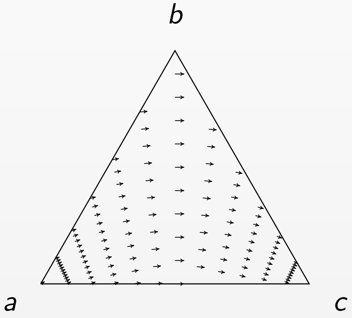

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Theorem 1: If $|U| \geq 4$, $f : \mathcal{D}^N \to \mathcal{R}$ is an anonymous Arrovian SWF, and $\mathcal{D}$ satisfies reasonable richness assumptions, then every $\succ \in \mathcal{D}$ is based on pairwise comparisons.

- Preferences over pure outcomes are unrestricted
- Preferences over all remaining outcomes are determined by preferences over pure outcomes

Characterization of Welfare Maximization

Theorem 2: If $|U| \geq 5$, $f\colon \mathcal{D}^N \to \mathcal{R}$ is an Arrovian SWF, and $\mathcal{D}$ is based on pairwise comparisons then, for all $R \in \mathcal{D}^N$,

$$\phi^{f(R)} = \sum_{i \in N} w_i \phi^{\succ_i},$$

for $w_i > 0$.

- The collective preferences over pure outcomes coincide with the majority relation

Remarks and Conclusions

- The theorems imply that individual preferences over pure outcomes cannot bare intensities but collective preferences can.
- The *finite support* assumption can be dropped under additional assumptions about Δ and $\mathcal{R}$.
- Given the individual preferences over pure outcomes, the collective preference between any two outcomes can be decided *efficiently*.
- The probabilistic social choice function that returns the maximal elements of the unique anonymous Arrovian SWF is known as *maximal lotteries*.
- Outcomes may be interpreted as lotteries, time shares, or allocations of divisible goods.